Stratovan

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• **Laplacian eigenfunctions:** The total force acting on a point with mass m_i is the sum of all spring forces and the gravitational force acting on point i (considering the sign +/– of spring forces):

$$f_1 = f_1^G + f_{1,2}^S$$
$$f_2 = f_2^G - f_{1,2}^S + f_{2,3}^S$$
$$f_3 = f_3^G - f_{2,3}^S$$

(unit: $\frac{kg \cdot m}{s^2} = 1N$)

Newton states that $f_i = m_i \ddot{x}_i$, $i = 1..3$, and the resulting equations are:

$$m_1 \ddot{x}_1 = -m_1 g - k_{1,2}(x_1 - x_2 - l_{1,2})$$
$$m_2 \ddot{x}_2 = -m_2 g + k_{1,2}(x_1 - x_2 - l_{1,2}) - k_{2,3}(x_2 - x_3 - l_{2,3})$$
$$m_3 \ddot{x}_3 = -m_3 g + k_{2,3}(x_2 - x_3 - l_{2,3})$$

These three second-order differential equations define a linear system:

$$\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} = \begin{bmatrix} -k_{1,2}/m_1 & k_{1,2}/m_1 & 0 \\ k_{1,2}/m_2 & (-k_{1,2}-k_{2,3})/m_2 & k_{2,3}/m_2 \\ 0 & k_{2,3}/m_3 & -k_{2,3}/m_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

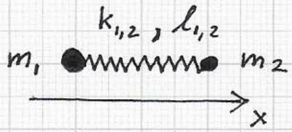
$$+ \begin{bmatrix} -g + (k_{1,2}\, l_{1,2})/m_1 \\ -g + (-k_{1,2}\, l_{1,2} + k_{2,3}\, l_{2,3})/m_2 \\ -g - (k_{2,3}\, l_{2,3})/m_3 \end{bmatrix}$$

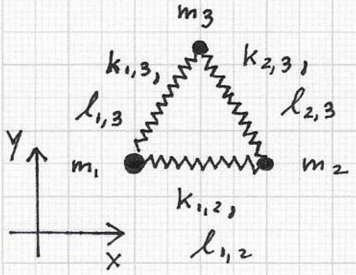
(units: m_i: kg; $x_i, l_{i,j}$: m; $g, \ddot{x}_i$: m/s^2; $k_{i,j}$: kg/s^2)

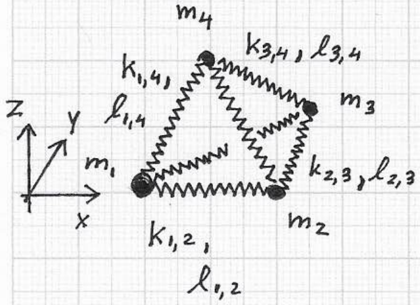
$$\Leftrightarrow \ddot{\mathbf{x}} = M\mathbf{x} + \mathbf{b}.$$

[RESULT: TRIGONOMETRIC FUNCTIONS!]

(This system can be solved analytically in closed form, given an initial condition for $t = t_0$, i.e., $x_1(t_0)$, $x_2(t_0)$ and $x_3(t_0)$.)



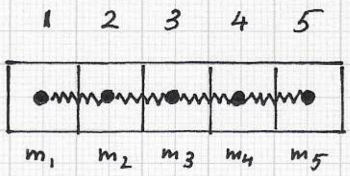




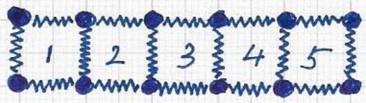
Simple mass-spring systems of dimensions one, two and three. Mass points m_i and m_j are connected by a spring with spring constant $k_{i,j}$ and equilibrium length $l_{i,j}$. The three simple systems are in equilibrium state, and there is no movement of any mass point relative to any of the other points.

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions:



Establishing a mass-spring model for a segment consisting of five pixels with masses m_i, $i=1...5$, and four identical springs that connect the centers of unit neighbor pixels. All springs have unit length $l_{i,i+1}=1$ in equilibrium state and have spring constants $k_{i,i+1}=1$, $i=1...4$.



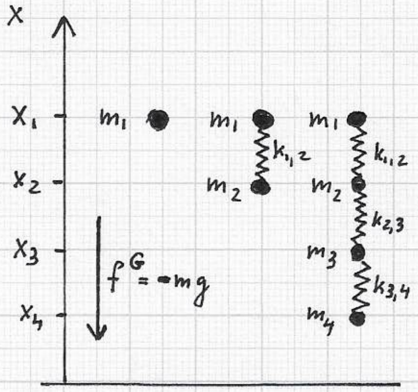
"Dual" mass-spring model. Instead of connecting centers of pixel neighbor and using these center-center edges to define springs, one can use the edges of pixels (edges of voxels in the 3D case) to define springs. Masses at pixel (voxel) corners can be assigned by using weighted averaging of pixel (voxel) masses in the respective neighborhood. This dual model could be advantageous for very small and very thin segments.

Note: Mass-spring models are commonly used in solid state physics to model, for example, crystal structures and compute (approximations of) eigenmode/eigenfrequency behavior. In our application – feature computation for data classification – we are NOT interested in a "high-fidelity computation of eigenfrequencies" of a 3D voxel segment; we merely employ a mass-spring model for a segment to devise a linear system of second-order partial differential equations leading to an eigenvalue problem qualitatively similar to that of a realistic physics-based computation. (For example, we can set all spring lengths and constants to one for a unit voxel segment and considering the at most six face-neighbors of a voxel; and we can neglect gravitation.)

Note: Eigenvalues/-vectors of the Helmholtz equation are known to support surface shape classification (Reuter et al., Laplace-Beltrami spectra as "shape-DNA"..., Computer-Aided Design 38). They are promising for inhomog. 3D solid classification.

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions:



ground level

Simple 1-, 2-, 4-point mass-spring models for points 1,…,4 with masses $m_1, \dots, m_4$. These additional examples serve the purpose of understanding the general nature of the eigenvalue matrix problem of an n-point mass-spring model.

These n-point models are merely discrete, finite approximations of a continuous, "infinite-dimensional problem," i.e., the dynamic, elastic behavior of a vertically positioned "rod."

Since this modeling approach is discrete and finite (n dimensions), one can only compute a finite number of eigenvalues and eigenfrequencies.

⇒ The no. of voxels of a segment defines the no. of the segment's eigenmodes.

To obtain a better understanding of the general n-point mass-spring model (and its governing differential equations), where points are connected by springs in a "linear chain-like fashion," we also consider the models for one, two and four points (see left figure).

1-point model:

$$f_1 = f_1^G$$

$$m_1 \ddot{x}_1 = -m_1 g$$

$$(\ddot{x}_1) = (0)\cdot(x_1) + (-g)$$

2-point model:

$$f_1 = f_1^G + f_{1,2}^S$$

$$f_2 = f_2^G - f_{1,2}^S$$

$$m_1 \ddot{x}_1 = -m_1 g - k_{1,2}(x_1 - x_2 - l_{1,2})$$

$$m_2 \ddot{x}_2 = -m_2 g + k_{1,2}(x_1 - x_2 - l_{1,2})$$

$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = \begin{pmatrix} -k_{1,2}/m_1 & k_{1,2}/m_1 \\ k_{1,2}/m_2 & -k_{1,2}/m_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} -g + (k_{1,2}\, l_{1,2})/m_1 \\ -g - (k_{1,2}\, l_{1,2})/m_2 \end{pmatrix}$$

4-point model:

$$f_1 = f_1^G + f_{1,2}^S$$

$$f_2 = f_2^G - f_{1,2}^S + f_{2,3}^S$$

$$f_3 = f_3^G - f_{2,3}^S + f_{3,4}^S$$

$$f_4 = f_4^G - f_{3,4}^S$$

OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions: ... 4-point model:

$$\begin{bmatrix}\ddot{x}_1\\ \ddot{x}_2\\ \ddot{x}_3\\ \ddot{x}_4\end{bmatrix} = \begin{bmatrix} -k_{1,2}/m_1 & k_{1,2}/m_1 & 0 & 0 \\ k_{1,2}/m_2 & (-k_{1,2}-k_{2,3})/m_2 & k_{2,3}/m_2 & 0 \\ 0 & k_{2,3}/m_3 & (-k_{2,3}-k_{3,4})/m_3 & k_{3,4}/m_3 \\ 0 & 0 & k_{3,4}/m_4 & -k_{3,4}/m_4 \end{bmatrix} \begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix}$$

$$+ \begin{bmatrix} -g + (k_{1,2}\, l_{1,2})/m_1 \\ -g + (-k_{1,2}\, l_{1,2} + k_{2,3}\, l_{2,3})/m_2 \\ -g + (-k_{2,3}\, l_{2,3} + k_{3,4}\, l_{3,4})/m_3 \\ -g + (-k_{3,4}\, l_{3,4})/m_4 \end{bmatrix}.$$

Example:

$g = 0\,[N]$

$m_i = 1\,[kg]$

$l_{i,i+1} = 1\,[m]$

$k_{i,i+1} = 1\,[kg/s^2]$

$\Rightarrow \omega_{i,i+1}^2 = \bar{\omega}_{i,i+1}^2 = 1\,[1/s^2]$

resulting system:

$$\begin{bmatrix}\ddot{x}_1\\ \ddot{x}_2\\ \ddot{x}_3\\ \ddot{x}_4\end{bmatrix} = \begin{bmatrix} -1 & 1 & & \\ 1 & -2 & 1 & \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} + \begin{bmatrix}1\\0\\0\\-1\end{bmatrix}$$

$\cdot\ \divideontimes$

$$\left[\frac{m}{s^2}\right] = \left[\frac{1}{s^2}\right]\cdot[m] + \left[m\cdot\frac{1}{s^2}\right]$$

The quantity $\omega = \sqrt{k/m}$ is the (angular) frequency (of the harmonic oscillator);

$$\omega = \sqrt{k/m} \Rightarrow \omega^2 = k/m.$$

Using the notation $\omega_{i,i+1}^2 = k_{i,i+1}/m_i$ and $\bar{\omega}_{i,i+1}^2 = k_{i,i+1}/m_{i+1}$, the 4-point system becomes

$$\begin{bmatrix}\ddot{x}_1\\ \ddot{x}_2\\ \ddot{x}_3\\ \ddot{x}_4\end{bmatrix} = \begin{bmatrix} -\omega_{1,2}^2 & \omega_{1,2}^2 & 0 & 0 \\ \bar{\omega}_{1,2}^2 & -\bar{\omega}_{1,2}^2-\omega_{2,3}^2 & \omega_{2,3}^2 & 0 \\ 0 & \bar{\omega}_{2,3}^2 & -\bar{\omega}_{2,3}^2-\omega_{3,4}^2 & \omega_{3,4}^2 \\ 0 & 0 & \bar{\omega}_{3,4}^2 & -\bar{\omega}_{3,4}^2 \end{bmatrix} \begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix}$$

$$+ \begin{bmatrix} -g + l_{1,2}\,\omega_{1,2}^2 \\ -g - l_{1,2}\,\bar{\omega}_{1,2}^2 + l_{2,3}\,\omega_{2,3}^2 \\ -g - l_{2,3}\,\bar{\omega}_{2,3}^2 + l_{3,4}\,\omega_{3,4}^2 \\ -g - l_{3,4}\,\bar{\omega}_{3,4}^2 \end{bmatrix}.$$

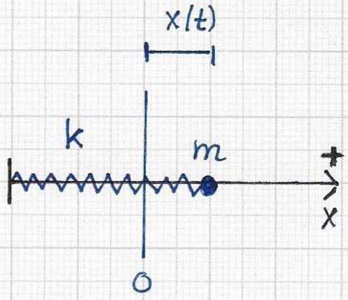
In matrix notation, and including the time dimension, the system can be written in matrix notation as

$$\ddot{\mathbb{x}}(t) = M\,\mathbb{x}(t) + \mathbb{b}.$$

An initial condition for $t = t_0$ defines $\mathbb{x}_0 = \mathbb{x}(t_0)$ and determines the solution of the system. (M is a tridiagonal matrix!)
In fact, TWO initial conditions are needed to define the solution of a SECOND-order differential equation.

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions: Merely for illustrative purposes, we briefly describe the (analytical) solution method for solving the linear system of second-order (coupled) differential equations for the simplified harmonic oscillator. (We do not consider gravitational or friction forces, since our segment/object classification scenario does not require them.)



Harmonic oscillator: 1 mass, 1 spring.

1-point homogeneous case:

$$m\ddot{x}(t) = -k\,x(t) \qquad \Big|\ \text{approach: } x(t) = e^{\lambda t}$$

$$m\lambda^2 e^{\lambda t} = -k\,e^{\lambda t} \qquad \Rightarrow \ddot{x} = \lambda^2 e^{\lambda t}$$

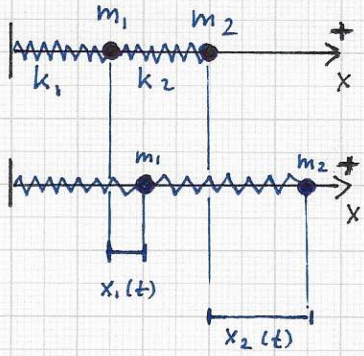
$$\lambda^2 = -k/m = -\omega^2$$

$$\Rightarrow \lambda_{1/2} = +/- \sqrt{-\omega^2}\text{, i.e., } \lambda_1 = i\omega \text{ and } \lambda_2 = -i\omega$$

$\Rightarrow$ general solution via superposition:

$$x(t) = A e^{\lambda_1 t} + B e^{\lambda_2 t} \quad (\text{or: } x(t) = A\cos(\omega t) + B\sin(\omega t))$$

$$\Big|\ \text{initial conditions } x_0 = x(t_0),\ \dot{x}_0 = \dot{x}(t_0) \Rightarrow A, B$$



Coupled harmonic oscillator: 2 masses, 2 springs.

2-point homogeneous case:

$$m_1\ddot{x}_1 = -k_1x_1 + k_2(x_2 - x_1)$$
$$m_2\ddot{x}_2 = -k_2(x_2 - x_1)$$

$$\Leftrightarrow \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = \begin{pmatrix} -k_1-k_2 & k_2 \\ k_2 & -k_2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$M\ddot{\mathbb{x}} = K\mathbb{x}$$

$$\Big|\ \text{approach: } \mathbb{x} = \mathbb{e}\,e^{\lambda t} \Rightarrow \ddot{\mathbb{x}} = \lambda^2 \mathbb{e}\,e^{\lambda t}$$

$$\Rightarrow M\lambda^2\mathbb{e} = K\mathbb{e} \qquad (\mathbb{e} = \text{"eigenvector"})$$

$$\Rightarrow (M\lambda^2 - K)\mathbb{e} = \mathbb{0}$$

$$\Rightarrow \det(M\lambda^2 - K) = 0 \qquad (\Rightarrow \text{eigenvalues})$$

$\Rightarrow$ eigenvalues, eigenvectors, general solution, specify $\mathbb{x}_0$ and $\dot{\mathbb{x}}_0$.

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