Stratovan

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

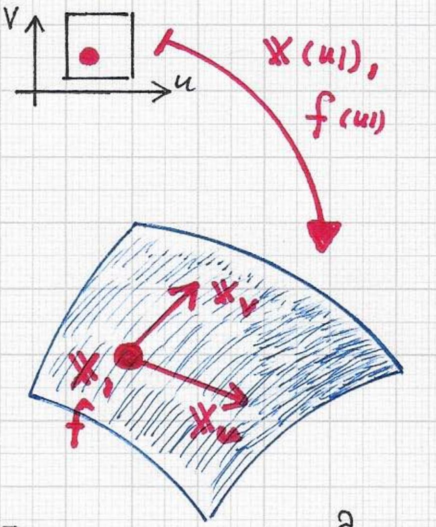
• Laplacian eigenfunctions:

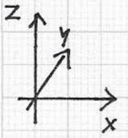
Prior to using the Laplacian operator Δ to compute Laplacian eigenvalues in the discrete, functional, trivariate setting involving a simple unit voxel grid, it is important to note the differences between our driving problem (computing Laplacian eigenvalue features for unit voxel grids with associated mass values) and Laplace-Beltrami eigenvalues that arise when dealing with a function defined on a curved, 2D manifold domain (or curved, 3D manifold domain) embedded in 3D (4D) space. We sketch the Laplace-Beltrami operator for a function f on a curved parametric surface $\mathbf{x}(u,v)$.

Beltrami defines the SECOND differential parameter of $f = f(u,v)$ in terms of the curvilinear coordinates u and v of a parametric surface $\mathbf{x} = \mathbf{x}(u,v)$:

$$\Delta f = \frac{1}{L}\Big(\big((Gf_u - Ff_v)/L\big)_u + \big((Ef_v - Ff_u)/L\big)_v\Big)/L.$$

The "subscripts" u and v refer to differentiation with respect to u and v, respectively. Further, $E = \langle \mathbf{x}_u, \mathbf{x}_u \rangle$, $F = \langle \mathbf{x}_u, \mathbf{x}_v \rangle$, $G = \langle \mathbf{x}_v, \mathbf{x}_v \rangle$, and $L = (EG - F^2)^{1/2}$. (Dirk Struik, Lectures on Classical Differential Geometry)





$\mathbf{x}_u = \frac{\partial}{\partial u}\mathbf{x}(u,v)$

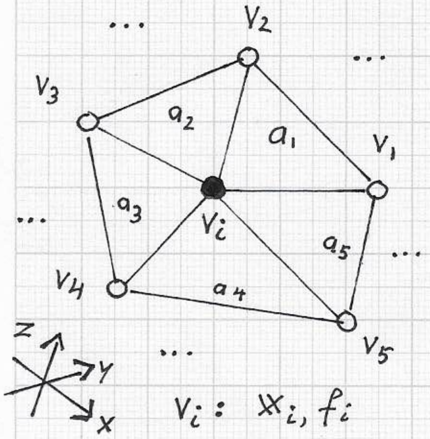
$\mathbf{x}_v = \frac{\partial}{\partial v}\mathbf{x}(u,v)$

Parametric surface setting for generalized Laplace-Beltrami operator. The function f to be analyzed has a curved surface as its domain; thus both the surface $\mathbf{x}$ and f are functions of u and v, i.e., $\mathbf{x} = \mathbf{x}(u,v)$ and $f = f(u,v)$.

Since f is a "function on a curved surface," the local differential geometry of the surface at a point $\mathbf{x}$ must be determined and used to compute the generalized Δf value at $\mathbf{x}$. As the Δ operator is generally defined as $\Delta f = \operatorname{div} \operatorname{grad} f$, one must use the partial derivatives $\mathbf{x}_u$ and $\mathbf{x}_v$ at $\mathbf{x}$ and use the partial derivative vectors to define the "grad" field subjected to the "div" operator.

OBJECT AND MATERIAL EIGENFUNCTIONS - Cont'd.

• Laplacian eigenfunctions:

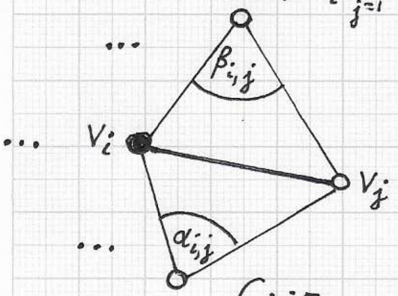


v_i : $\mathbb{x}_i, f_i$

v_j : $\mathbb{x}_j, f_j$, $j = 1 \ldots 5$

a_j: triangle area

$\Rightarrow A_i = \sum_{j=1}^{5} a_j$



$C_{i,j} = \cot \alpha_{i,j} + \cot \beta_{i,j}$

Local data needed to estimate Δf at a triangle mesh vertex using a commonly used scheme that consider geometrical information (areas and angles) and functional data (f-values at vertices).

Formulae must be able to handle mes boundaries.

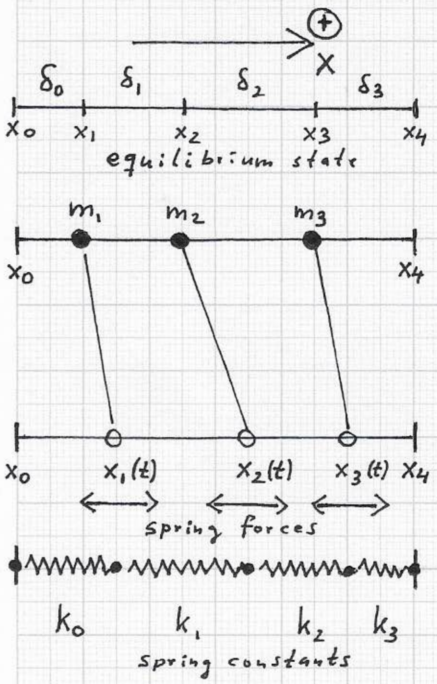
Beltrami's definition of the second differential parameter of a function $f = f(\mathbb{u})$ on a curved surface, leading to the general Laplace-Beltrami operator, can be used to analyze a function f when it is known only at points $\mathbb{x}_i$ in 3D space with associated function values f_i; commonly, a simple triangle mesh is used to define a continuous, piecewise linear, domain for the function to be analyzed. Given a set $\{\mathbb{x}_i, f_i\}$ and a mesh, one can locally define many meaningful estimates of $\Delta f(\mathbb{x}_i)$. In this case, a parametric map $\mathbb{u} \mapsto \mathbb{x}(\mathbb{u})$, $\mathbb{u} \mapsto f(\mathbb{u})$ is not known. Triangle meshes are commonly used in many applications, and various approximation schemes exist to estimate Δf at mesh vertices v_i when f-values are known only at these vertices. One standard approximation of $\Delta f(v_i)$ is:

$$\Delta(v_i) = \frac{3}{2A_i} \sum_{j=1}^{5} C_{i,j} (f_i - f_j).$$

This approximation formula can be derived by viewing f as a piecewise-defined continuous function on the triangle mesh. (Keenan Crane et al., Digital Geometry Processing with Discrete Exterior Calculus, course notes, ACM SIGGRAPH 2013)

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions: We are merely interested in the discrete, functional case when considering Δf. More specifically, we only have to define and compute Δf-values for centers (or corners) of voxels with associated masses defining a segment/object/material in a 3D scan. Once we have Δf-values for all voxel centers (or corners), we can set up the eigenvalue problem $\Delta f = -\lambda f$. It is helpful to consider the 1D case of of a harmonic oscillator with point masses connected by springs as a linear chain (see illustration, left). This is the set-up:



1D example of a "mass-spring model used to derive the eigenvalue problem that defines the oscillating systems eigenmodes/-frequencies.

- 3 point masses m_1, m_2, m_3 are given.
- The 3 points' positions in rest state are x_1, x_2, x_3.
- The "wall end points" are x_0 and x_4.
- The rest state spacing is given by
  $$\delta_i = x_{i+1} - x_i \quad , \quad i = 0 \ldots 3 .$$
- Point pairs x_i and x_{i+1} are connected by a spring with spring constants k_i.
- The time-dependent positions of the three "free mass points" are $x_i(t)$, $i = 1 \ldots 3$.
- Newton's and Hooke's laws describe the systems behavior via second-order differential equations: $m\ddot{x}(t) = -kx(t)$.

OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions: Considering the illustration and parameter explanations from the previous page, the system must satisfy these equations:

Derivation of linear system of second-order differential equations describing the mass-spring system of point masses on a "linear chain"

$$m_1 \ddot{x}_1(t) = -k_0\left((x_1(t)-x_0(t))-\delta_0\right) + k_1\left((x_2(t)-x_1(t))-\delta_1\right)$$
$$= k_0 x_0(t) + (-k_0-k_1)x_1(t) + k_1 x_2(t) + k_0\delta_0 - k_1\delta_1$$

$$m_2 \ddot{x}_2(t) = -k_1\left((x_2(t)-x_1(t))-\delta_1\right) + k_2\left((x_3(t)-x_2(t))-\delta_2\right)$$
$$= k_1 x_1(t) + (-k_1-k_2)x_2(t) + k_2 x_3(t) + k_1\delta_1 - k_2\delta_2$$

$$m_3 \ddot{x}_3(t) = -k_2\left((x_3(t)-x_2(t))-\delta_2\right) + k_3\left((x_4(t)-x_3(t))-\delta_3\right)$$
$$= k_2 x_2(t) + (-k_2-k_3)x_3(t) + k_3 x_4(t) + k_2\delta_2 - k_3\delta_3$$

$\Rightarrow$

$$m_i \ddot{x}_i(t) = k_{i-1}x_{i-1}(t) + (-k_{i-1}-k_i)x_i(t) + k_i x_{i+1}(t) + k_{i-1}\delta_{i-1} - k_i\delta_i \quad, i=1\ldots3,$$

where $x_0(t)=x_0$, $x_4(t)=x_4$.

The resulting inhomogeneous system is solved with initial conditions $\mathbb{x}(t_0)=\mathbb{x}_0$ and $\dot{\mathbb{x}}(t_0)=\dot{\mathbb{x}}_0$.

In matrix notation, this linear system is:

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}\begin{bmatrix} \ddot{x}_1(t) \\ \ddot{x}_2(t) \\ \ddot{x}_3(t) \end{bmatrix} = \begin{bmatrix} -k_0-k_1 & k_1 & 0 \\ k_1 & -k_1-k_2 & k_2 \\ 0 & k_2 & -k_2-k_3 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} k_0x_0+k_0\delta_0-k_1\delta_1 \\ k_1\delta_1-k_2\delta_2 \\ k_3x_4+k_2\delta_2-k_3\delta_3 \end{bmatrix}.$$

$$M \quad \ddot{\mathbb{x}}(t) = \quad K \quad \mathbb{x}(t) + \quad \mathbb{b}$$

OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions: In our application – feature value computation for volumetric object segments defined by voxels with voxel-specific masses – we can set all spring constant values k_i and all spacing values δ_i to 1. In this setting, the inhomogeneous system is

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \ddot{\mathbb{x}}(t) = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix} \mathbb{x}(t) + \begin{bmatrix} x_0 \\ 0 \\ x_4 \end{bmatrix}.$$

$$\underset{\text{(mass)}}{M} \; \ddot{\mathbb{x}}(t) = \underset{\text{(stiffness)}}{K} \; \mathbb{x}(t) + \mathbb{b}$$

Eigenvectors:

$\mathbb{e}_1 = (e_1^x, e_1^y, e_1^z)^T$

$\mathbb{e}_2 = (e_2^x, e_2^y, e_2^z)^T$

$\mathbb{e}_3 = (e_3^x, e_3^y, e_3^z)^T$

Eigenvalues:

$\lambda_1 = \ldots, \lambda_2 = \ldots, \lambda_3 = \ldots$

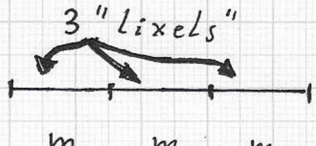
The actual eigenvalue problem we are interested in concerns the homogeneous system:

$$\ddot{\mathbb{x}}(t) = M^{-1} K \, \mathbb{x}(t)$$

$$\Longleftrightarrow \quad \ddot{\mathbb{x}}(t) - M^{-1} K \, \mathbb{x}(t) = \mathbb{0}.$$

Considering our example with 3 mass points, the matrix we must analyze is $M^{-1}K$:

$$M^{-1}K = \begin{pmatrix} 1/m_1 & 0 & 0 \\ 0 & 1/m_2 & 0 \\ 0 & 0 & 1/m_3 \end{pmatrix} \begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{pmatrix} = \begin{pmatrix} -2/m_1 & 1/m_1 & 0 \\ 1/m_2 & -2/m_2 & 1/m_2 \\ 0 & 1/m_3 & -2/m_3 \end{pmatrix}.$$



$\Rightarrow$ 3 eigenvalues for 3-lixel segment

In this simple "1D segment scenario," one associates masses m_1, m_2 and m_3 with line elements ("lixels") of length 1, and the 3 eigenvalues define a feature for this segment. (Indexing of and connectivity between mass points $\Rightarrow$ K is tridiagonal.)