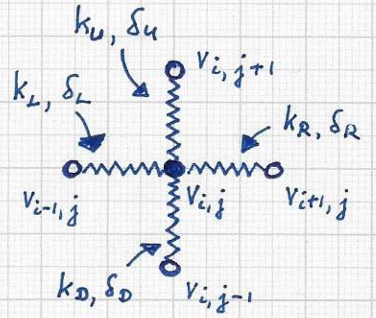
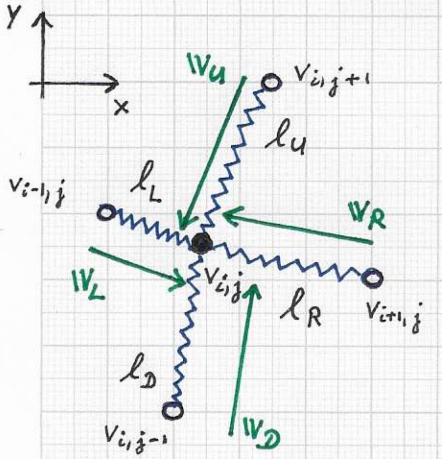
■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions:





Parameters needed and notation used to describe the local stencil of a 2D, discretized oscillating system. Top: Rest state of system. Bottom: Intermediate state of oscillating system. (Boundaries require special case treatment.)

The non-rest-state spring lengths are l_L, l_R, l_U, l_D. Relevant difference vectors between the centre point position and neighbor point positions are $\mathbb{v}_L = \mathbb{x}_{i,j} - \mathbb{x}_{i-1,j}$, ... $\mathbb{v}_D = \mathbb{x}_{i,j} - \mathbb{x}_{i,j-1}$. When normalizing these vectors one obtains $\overline{\mathbb{v}}_L, \dots, \overline{\mathbb{v}}_D$ of length one.

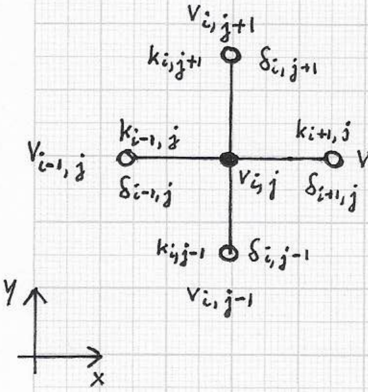
Concerning the 2D generalization, we only consider the case where (internal) mass points have four neighbor mass points. We use the notation shown in the illustrations: A vertex in the mass spring model is called $v_{i,j}$; it has the associated position $\mathbb{x}_{i,j}$ and mass $m_{i,j}$. The four neighbors of $v_{i,j}$ are $v_{i-1,j}$, $v_{i+1,j}$, $v_{i,j-1}$ and $v_{i,j+1}$, called **L**eft, **R**ight, **D**own and **U**p neighbors, respectively. The four springs connecting $v_{i,j}$ with its four neighbors have spring constants k_L, k_R, k_D and k_U; the "rest state"/"equilibrium state" lengths of the springs are $\delta_L, \delta_R, \delta_D$ and δ_U. (The top picture depicts the rest state.) When the system oscillates, the positions of the vertices change and springs expand or compress. (The bottom picture depicts such an intermediate state.) The equations describing the system's behavior are:

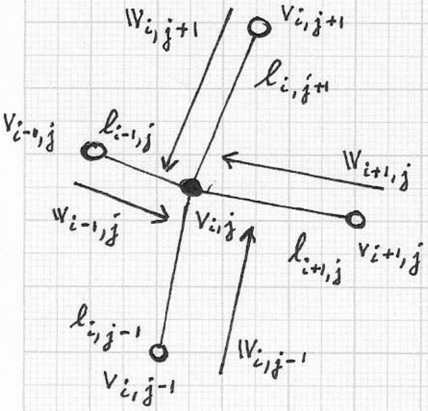
$$
\begin{aligned}
m_{i,j}\, \ddot{\mathbb{x}}_{i,j}(t) &= -k_L\,(l_L - \delta_L)\,\overline{\mathbb{v}}_L - k_R\,(l_R - \delta_R)\,\overline{\mathbb{v}}_R \\
&\quad - k_D\,(l_D - \delta_D)\,\overline{\mathbb{v}}_D - k_U\,(l_U - \delta_U)\,\overline{\mathbb{v}}_U \\
&= -k_L\,\left(\|\mathbb{x}_{i,j}(t) - \mathbb{x}_{i-1,j}(t)\| - \delta_L\right) \\
&\quad \cdot \left(\left(\mathbb{x}_{i,j}(t) - \mathbb{x}_{i-1,j}(t)\right) / \|\mathbb{x}_{i,j}(t) - \mathbb{x}_{i-1,j}(t)\|\right) \\
&\quad - k_R \cdot \ldots - k_D \cdot \ldots - k_U \cdot \ldots \quad .
\end{aligned}
$$

(The number of vertices $v_{i,j}$ defines the number of equations.)

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• **Laplacian eigenfunctions:** It is possible to "simplify" the notation by using just double-indices (i,j) for all parameters, as shown in the illustrations (left). Spring-associated information is simply indexed with the indices of those neighbor vertices of vertex $v_{i,j}$ that are connected to $v_{i,j}$. The set N associated with vertex $v_{i,j}$ stores the four index tuples of $v_{i,j}$'s four neighbors, i.e., $N = \{(i-1,j), (i+1,j), (i,j-1), (i,j+1)\}$. The equations become:





Using a consistent and simplified indexing scheme for the five-vertex stencil associated with $v_{i,j}$.

$$m_{i,j}\, \ddot{\mathbb{x}}_{i,j}(t) = -\sum_{(I,J)\in N} k_{I,J} \left(l_{I,J} - \delta_{I,J} \right) \mathbb{w}_{I,J}$$

$$= -\sum_{(I,J)\in N} k_{I,J} \left(\| \mathbb{x}_{i,j}(t) - \mathbb{x}_{I,J}(t) \| - \delta_{I,J} \right) \cdot \frac{\mathbb{x}_{i,j}(t) - \mathbb{x}_{I,J}(t)}{\| \mathbb{x}_{i,j}(t) - \mathbb{x}_{I,J}(t) \|} .$$

Using "boldface notation" for multi-indices, i.e., $\mathbb{i} = (i,j)$ and $\mathbb{I} = (I,J)$, the equations are:

$$m_{\mathbb{i}}\, \ddot{\mathbb{x}}_{\mathbb{i}}(t) = -\sum_{\mathbb{I}\in N} k_{\mathbb{I}} \left(l_{\mathbb{I}} - \delta_{\mathbb{I}} \right) \mathbb{w}_{\mathbb{I}} .$$

Since $\ddot{\mathbb{x}}_{\mathbb{i}}(t) = \begin{pmatrix} \ddot{x}_{\mathbb{i}}(t) \\ \ddot{y}_{\mathbb{i}}(t) \end{pmatrix}$ and $\mathbb{w}_{\mathbb{I}} = \begin{pmatrix} \bar{w}^x_{\mathbb{I}} \\ \bar{w}^y_{\mathbb{I}} \end{pmatrix}$, one can write

$$m_{\mathbb{i}} \begin{pmatrix} \ddot{x}_{\mathbb{i}}(t) \\ \ddot{y}_{\mathbb{i}}(t) \end{pmatrix} = -\sum_{\mathbb{I}\in N} k_{\mathbb{I}} \left(\left\| \begin{pmatrix} x_{\mathbb{i}}(t) \\ y_{\mathbb{i}}(t) \end{pmatrix} - \begin{pmatrix} x_{\mathbb{I}}(t) \\ y_{\mathbb{I}}(t) \end{pmatrix} \right\| - \delta_{\mathbb{I}} \right) \cdot \frac{\begin{pmatrix} x_{\mathbb{i}}(t) \\ y_{\mathbb{i}}(t) \end{pmatrix} - \begin{pmatrix} x_{\mathbb{I}}(t) \\ y_{\mathbb{I}}(t) \end{pmatrix}}{\left\| \begin{pmatrix} x_{\mathbb{i}}(t) \\ y_{\mathbb{i}}(t) \end{pmatrix} - \begin{pmatrix} x_{\mathbb{I}}(t) \\ y_{\mathbb{I}}(t) \end{pmatrix} \right\|} .$$

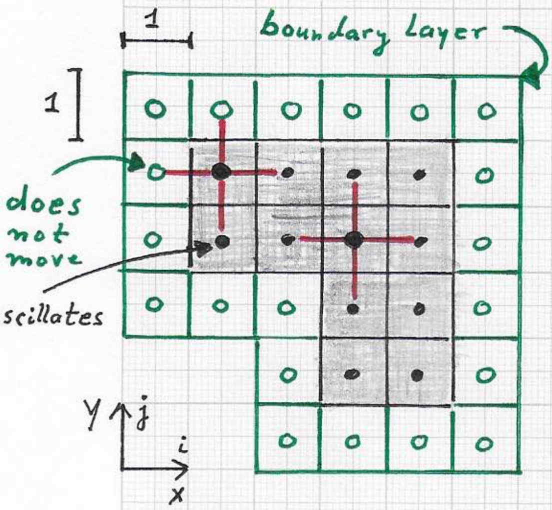
"Two systems" of differential equations result: one system for the x-dimension, one for the y-dimension!

→ "Quasi-superposition" of a longitudinal wave (in x) and transversal wave (in y)!

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

•Laplacian eigenfunctions: One can also write these equations as

$$m_{\underline{i}}\, \ddot{\mathbf{x}}_{\underline{i}}(t) = -\sum_{\underline{I} \in N} k_{\underline{I}} \left(\mathbf{x}_{\underline{i}}(t) - \mathbf{x}_{\underline{I}}(t) \right) - \delta_{\underline{I}} \,/\, \| \mathbf{x}_{\underline{i}}(t) - \mathbf{x}_{\underline{I}}(t) \| .$$



2D example: segment (shaded pixels) and surrounding "ghost pixels" (= boundary layer). Two segment pixels are emphasized, one being connected to four other segment pixels and one being connected to two segment and two boundary layer pixels.

In our application, one can use one spring constant ($k_{\underline{i}} = k$) and use a constant rest state spring length $\delta_{\underline{i}} = 1$, given an image consisting of unit square pixels.

This model of a 2D harmonic oscillator system would induce a behavior of the 2D segment's mass points (= segment pixel center points) that is characterized by periodic, oscillating movements of the mass points (inside the pixels' unit regions with a proper k-value).

These equation describe the dynamic behavior of the mass points $\mathbf{x}_{\underline{i}}$ of a 2D mass spring system, i.e., the time-dependent function $\mathbf{x}_{\underline{i}}(t) = (x_{\underline{i}}(t), y_{\underline{i}}(t))^T$. Since $m_{\underline{i}}\, \ddot{\mathbf{x}}_{\underline{i}}(t) = -\sum_{\underline{I} \in N} \ldots$, the acceleration of this mass point is $\ddot{\mathbf{x}}_{\underline{i}}(t) = -\frac{1}{m_{\underline{i}}} \sum_{\underline{I} \in N} \ldots$.

"If one were interested in the dynamic behavior of the coupled movements of all mass points defining one 2D segment," one would have to use a numerical integration method to approximate the positions $\mathbf{x}_{\underline{i}}$ for time points $t_0, t_1, t_2, \ldots$ The most appropriate method for integration in this setting is the FOURTH-ORDER RUNGE-KUTTA METHOD WITH ADAPTIVE TIME STEP SIZE. (Collectively the mass points perform a complex superposition of periodic, oscillating "particle movements." Solid state physicists study atomistic behavior of crystals with mass spring models...) For integration, one must specify initial conditions, i.e., $\mathbf{x}_{\underline{i}}(t_0) = \ldots$ and $\dot{\mathbf{x}}_{\underline{i}}(t_0)$.

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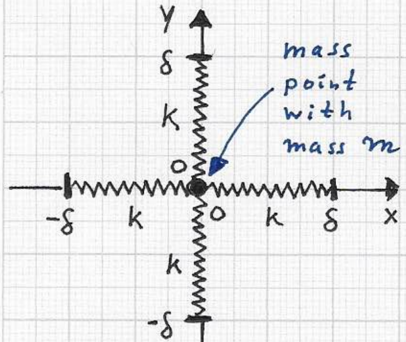
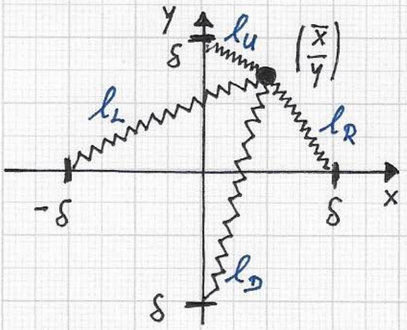
Bernd Hamann

9/16/2021
Stratovan

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions:





Simple 2D scenario of a single mass point with mass m permitting the "**de-coupling**" of the differential equations describing the point's movement in x-direction ("longitudinal" direction) and y-direction ("transversal" direction).

One can obtain analytical, closed-form solutions for the coordinate functions $x(t)$ and $y(t)$, i.e., trigonometric functions, describing the point's periodic, oscillating movement.

Just for examplary purposes, we consider the case of a single mass point connected via springs to four fixed, not movable wall points. The left illustrations depict the rest/equilibrium state (top) and an intermediate state of the mass point's movement. All springs have spring constant k and have rest state length δ. In rest state, the mass point has coordinates $\binom{0}{0}$, and the four fixed neighbor points are $\binom{-\delta}{0}$, $\binom{\delta}{0}$, $\binom{0}{-\delta}$, $\binom{0}{\delta}$. WE ASSUME THAT THE MASS POINT ONLY MOVES A VERY SMALL DISTANCE relative to the value of δ, i.e., $\bar{x}, \bar{y} \ll \delta$. Thus:

$$\ell_L = \| (\bar{x},\bar{y})^T - (-\delta,0)^T \| = \left((\bar{x}+\delta)^2 + \bar{y}^2 \right)^{1/2} \approx \bar{x}+\delta \quad ,$$

$$\ell_R = \| (\bar{x},\bar{y})^T - (\delta,0)^T \| = \left((\bar{x}-\delta)^2 + \bar{y}^2 \right)^{1/2} \approx \bar{x}-\delta \quad ,$$

$$\ell_D = \| (\bar{x},\bar{y})^T - (0,-\delta)^T \| = \left(\bar{x}^2 + (\bar{y}+\delta)^2 \right)^{1/2} \approx \bar{y}+\delta \quad ,$$

$$\ell_U = \| (\bar{x},\bar{y}) - (0,\delta)^T \| = \left(\bar{x}^2 + (\bar{y}-\delta)^2 \right)^{1/2} \approx \bar{y}-\delta \quad .$$

Therefore, the force components in x- and y-directions acting on the mass point are

$$f_x = -k(\ell_L - \delta) + k(-\delta - \ell_R) = -k\bar{x} - k\bar{x} = -2k\bar{x},$$

$$f_y = -k(\ell_D - \delta) + k(-\delta - \ell_U) = -k\bar{y} - k\bar{y} = -2k\bar{y}.$$

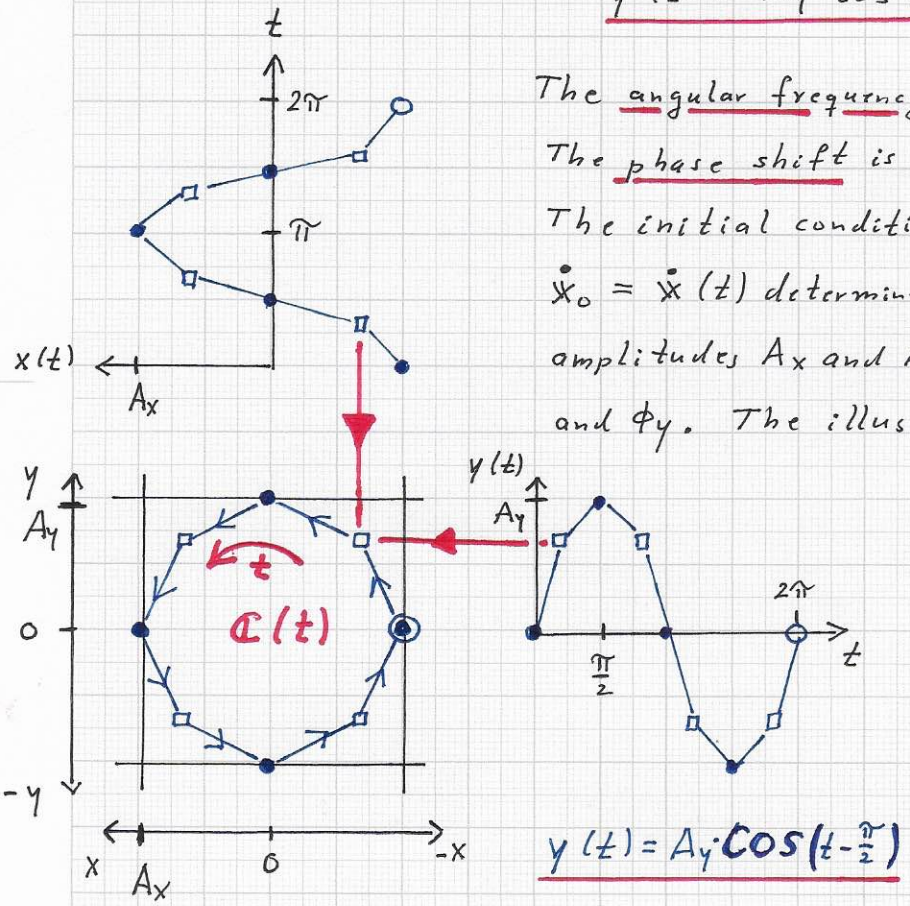
The resulting "de-coupled" differential equations describing the systems behavior are $m\ddot{x}(t) = -2k\,x(t)$ and $m\ddot{y}(t) = -2k\,y(t)$. ...

OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions: ... The closed-form solutions to these de-coupled differential equations are

$$x(t) = A_x \cos(\omega t + \phi_x) \,,$$
$$y(t) = A_y \cos(\omega t + \phi_y) \,.$$

$x(t) = A_x \cdot \cos(t - \pi)$



The angular frequency is $\omega = \sqrt{k/m}$. The phase shift is denoted by ϕ_x and ϕ_y. The initial conditions $\mathbb{x}_0 = \mathbb{x}(t_0)$ and $\dot{\mathbb{x}}_0 = \dot{\mathbb{x}}(t)$ determine the values of the amplitudes A_x and A_y and phase shifts ϕ_x and ϕ_y. The illustration shows an example result: The coordinate functions $x(t)$ and $y(t)$ define a circle $\mathbb{C}$ ($A_x = A_y$) as locus for the movement of the mass point ($\omega = 1$), i.e., $\mathbb{C}(t) = (x(t), y(t))^T$.

$y(t) = A_y \cdot \cos(t - \frac{\pi}{2})$

Simple example of the idealized oscillating behavior of a single mass point – moving on the circle $x^2 + y^2 = A_x^2$. If the mass point was "physical and visible," an observer would see the point orbiting the origin on a circle.

In other words, the resulting "trajectory or pathline of the mass point" is the superposition of the functions $x(t)$ and $y(t)$. The trajectory $\mathbb{C}(t)$ is commonly referred to as a **LISSAJOUS FIGURE** in physics.