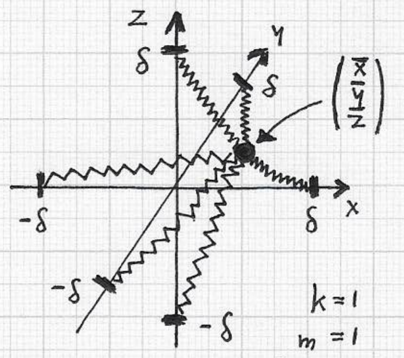
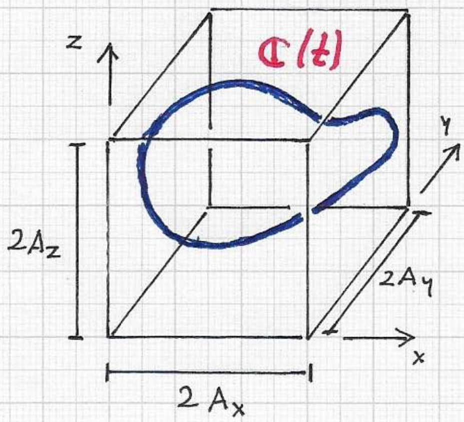
■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions: When extending the example to a single mass point, now connected by springs to six fixed neighbor points, and using the same simplified assumptions and parameter values, the mass point's coordinate functions (de-coupled) are



$$x(t) = A_x \cdot \cos(t + \phi_x),$$
$$y(t) = A_y \cdot \cos(t + \phi_y),$$
$$z(t) = A_z \cdot \cos(t + \phi_z).$$



The "path" of the mass point in 3D space is a curve $\mathbb{C}(t)$ inside a bounding box with edge lengths A_x, A_y and A_z (see left figures). In the case of a 3D volume segment consisting of voxels with associated masses, one can keep this simple one-mass-point model in mind – and think of the "voxel center points as mass points all performing a similar harmonically oscillating movement."

Simple 3D scenario of one mass point connected to six fixed neighbor points. Using the simplifications made in the 2D case, the resulting mass point's trajectory lies in a bounding box with edge lengths defined by the three amplitudes of the three coordinate functions.

While this time-dependent analysis and characterization is interesting and potentially valuable for segment characterization, we prefer a simpler, more "compact" characterization of a segment in terms of eigenvalues/-modes.

BH

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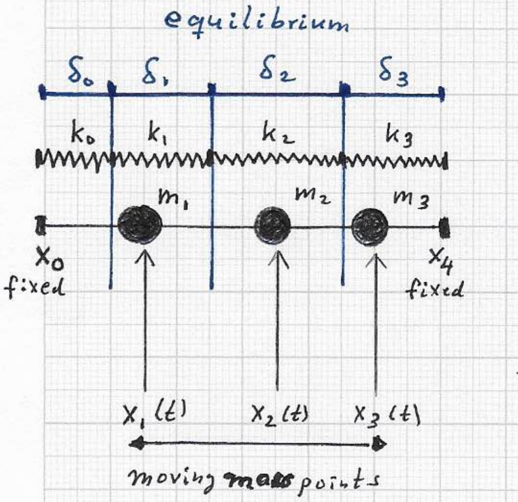
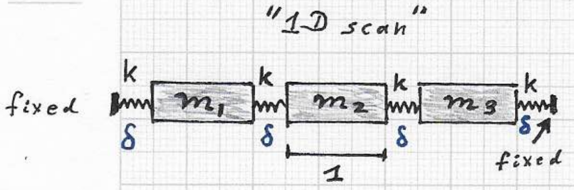
■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions:





Top: Model of three mass points defining a coupled harmonic oscillator system. Bottom: "1D scan" consisting of three unit-length line elements – "LIXELS" – with associated masses m_1, m_2, m_3; even though the lixels are directly, leaving no space between them, we "model their coupled oscillations" by placing "imaginary springs with spring constant k" between the lixels.

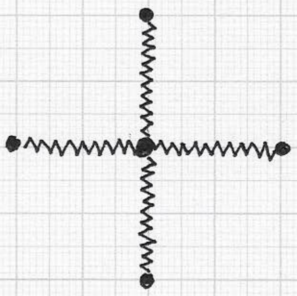
$$M^{-1}K = \begin{bmatrix} \frac{-k_0-k_1}{m_1} & \frac{k_1}{m_1} & 0 \\ \frac{k_1}{m_2} & \frac{-k_1-k_2}{m_2} & \frac{k_2}{m_2} \\ 0 & \frac{k_2}{m_3} & \frac{-k_2-k_3}{m_3} \end{bmatrix}$$

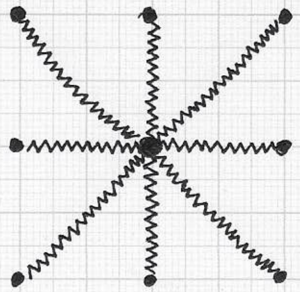
The figure shown left (top) shows the relevant parameters of a coupled system of three oscillators. (This example is also used on pages 3, 4 and 5, 9/11/21, 9/13/21 and 9/14/21, to discuss the eigen-value/-vector problem we are interested in.) It was shown that the linear system describing this system is of the form $M\ddot{\mathbf{x}}(t) = K\mathbf{x}(t) + \mathbf{b}$. The homogeneous part of this system is relevant for us: $\ddot{\mathbf{x}}(t) = M^{-1}K\,\mathbf{x}(t)$. $M^{-1}K$ is the product of the inverse of the mass matrix M and the stiffness matrix K with a structure that is defined by the connectivity of mass points as established by the connecting springs. We want to apply this mass-spring-model view directly to our application: We have unit-edge-length voxels of a segment in a 3D scan, having voxel-associated masses; we assume "imaginary springs between a segment's voxels." The simplest connectivity between voxels via springs is the six-neighbor connectivity (left, right, front, back, down, up), defining K.

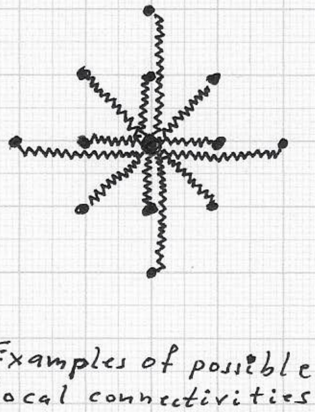
Eigenvalues/-vectors of $M^{-1}K$ define a segment's material and geometry signature – a SPECTRUM. !

OBJECT AND MATERIAL EIGENFUNCTIONS - Cont'd.

• Laplacian eigenfunctions:







Examples of possible local connectivities between a central mass point and a set of mass points in a local neighborhood. The spring connectivity defines the stiffness matrix $\mathbf{K}$ and therefore also the eigenvalues of $\mathbf{M}^{-1}\mathbf{K}$.

For example, when assuming that $m_i = 1$ and $k_i = 1$, $i = 1 \ldots 3$, for the scenario used on the previous page, one obtains eigenvalues

$$\lambda_1 = -2 - \sqrt{2} \quad , \quad \lambda_2 = -2 \quad , \quad \lambda_3 = \sqrt{2} - 2$$

and associated eigenvectors

$$\mathbf{e}_1 = (1, -\sqrt{2}, 1)^T, \ \mathbf{e}_2 = (-1, 0, 1)^T, \ \mathbf{e}_3 = (1, \sqrt{2}, 1)^T.$$

(The three eigenvectors are orthogonal to each other and define the "system-specific orthogonal basis!)

Since we will apply this physics-based model of coupled oscillators to the setting of "coupled oscillating voxels of a 3D segment," it seems desirable to discuss coupled oscillators on a high level, to provide some background of the mathematical concepts.

We first consider the case of n mass points with masses m_i moving/oscillating in just one dimension (x); the connectivity between mass points can be "general," i.e., any mass point i can be connected to a mass point j. The equations describing this case are: $m_i \, \ddot{x}_i(t) = -\sum_j k_{i,j} \, x_j(t)$, $i = 1 \ldots n$. One can use matrix notation and describe the coupled oscillator system compactly.

OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• **Laplacian eigenfunctions:** We write the mass and stiffness matrices as follows:

$$M = \begin{bmatrix} m_1 & & 0 \\ & \ddots & \\ 0 & & m_n \end{bmatrix}, \quad K = \begin{bmatrix} k_{1,1} & \cdots & k_{1,n} \\ \vdots & & \vdots \\ k_{n,1} & \cdots & k_{n,n} \end{bmatrix}.$$

Writing positions and accelerations of mass points as column vectors

$\mathbf{x}(t) = (x_1(t), \ldots, x_n(t))^T$ and $\ddot{\mathbf{x}}(t) = (\ddot{x}_1(t), \ldots, \ddot{x}_n(t))^T$,

the motions can be summarized as

$$M\,\ddot{\mathbf{x}}(t) = -K\,\mathbf{x}(t), \text{ i.e.,}$$

$$\ddot{\mathbf{x}}(t) = -M^{-1}K\,\mathbf{x}(t).$$

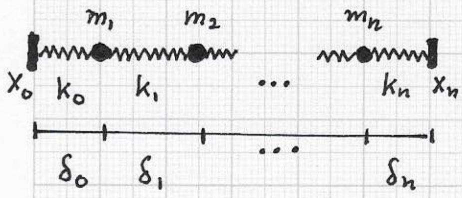
The eigenvalue problem we must solve is

$$M^{-1}K\,\Phi = \lambda\,\Phi = \omega^2\,\Phi.$$

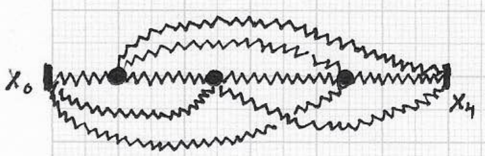
(The squared "eigenmode frequencies" ω_i are the resulting eigenvalues λ_i.)

In our application – characterizing a 3D segment consisting of voxels with associated masses – our mass points reside in 3D space and "oscillate" in x-, y- and z-direction. For situations where "mass points move away from their rest state positions only by very small amounts" one can effectively "de-couple" the points' $x(t)$, $y(t)$ and $z(t)$ movements and treat them independently (p. 9, 9/16/21).

equilibrium



Basic, simple connectivity model for n coupled mass points; each mass point i is connected (at most) to two other mass points, i.e., (i-1) and (i+1). As a consequence the matrices K and $M^{-1}K$ are tridiagonal matrices.



"Fully coupled," "fully connected" mass spring model. Since every mass point i is connected to every other mass point, the associated stiffness matrix K is a "full matrix."

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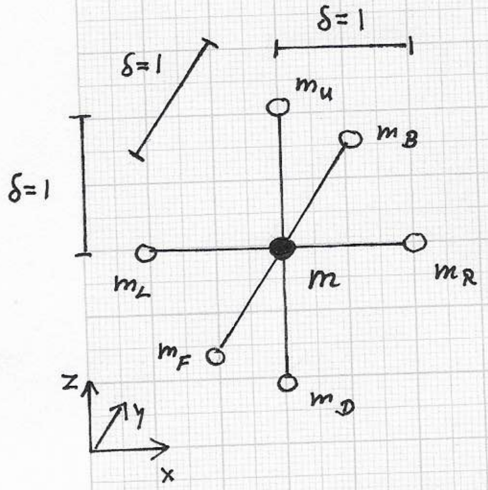
9/21/2021

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions: By de-coupling the "movements of a mass point in 3D space for very small-amplitude oscillations," the movements can be written as

$$\begin{bmatrix} | & | & | \\ \ddot{\mathbf{x}}(t) & \ddot{\mathbf{y}}(t) & \ddot{\mathbf{z}}(t) \\ | & | & | \end{bmatrix} = -M^{-1}K \begin{bmatrix} | & | & | \\ \mathbf{x}(t) & \mathbf{y}(t) & \mathbf{z}(t) \\ | & | & | \end{bmatrix},$$

$$\ddot{X}(t) = -M K X(t) .$$



Local internal segment stencil: center voxel with mass m has left, right, front, back, down and up neighbor voxels with masses m_L, m_R, m_F, m_B, m_D and m_U, respectively.

Since $\delta = 1$ and $k = 1$, the entries of the stiffness matrix **K** are defined entirely by the chosen six-neighbor connectivity; K is sparse.

The notation $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ is used for column vectors of all mass points' coordinates in x-, y-, and z-dimension.

FOR SIMPLICITY AND SYMMETRY WE ASSUME **THAT** "A MASS POINT IS THE CENTER OF A VOXEL WITH EDGE LENGTH ONE", WHERE THE VOXEL HAS "SPRING CONNECTIONS" ONLY WITH ITS SIX CLOSEST NEIGHBORS (left, right, front, back, down, up). We also assume that the rest state distances between every voxel's center and the centers of its (up to) six neighbor voxels is one ($\delta = 1$). Further, we use the same spring constant everywhere, i.e., $k_i = k = \mathbf{1}$. FOR OUR APPLICATION ONLY EIGENVALUES/-VECTORS OF $M^{-1}K$ MATTER, to define segment features.

~ BH