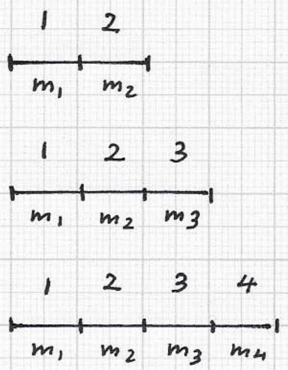
■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions:



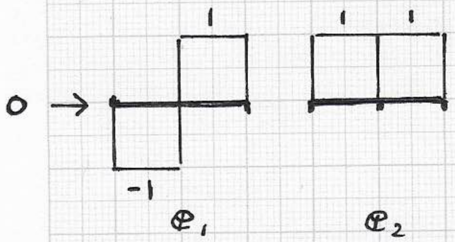
Simple 1D examples: two, three and four line elements, "lixels," with associated masses m_1, m_2, m_3 and m_4. The number of lixels defines the dimensionality of the eigenvalue problem — and the number of resulting eigenvectors/ eigenfrequencies/ eigenmodes/ eigenfunctions.

We present several 1D examples to better understand the general application of the approach. In the 1D setting, line elements ("lixels") of length one have an associated mass and have (at most) two neighbor lixels (left, right). We further simplify our initial examples by setting all mass values to one. We begin with the illustrated three cases: two, three and four lixels. For these cases the corresponding $M^{-1}K$ matrices are: $\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$, $\begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix}$, $\begin{bmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}$.

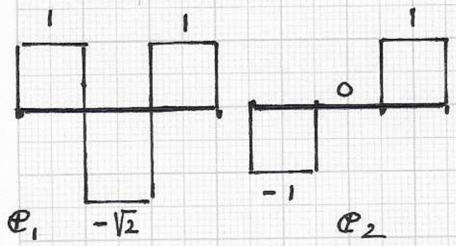
(One can view the triple $(1,-2,1)$ defining the central difference formula for the second derivative of a function using three function values f_{i-1}, f_i, and f_{i+1} with weights 1, -2, and 1; or one can view the triple $(1,-2,1)$ as a way to encode graph information: the center lixel has two outgoing edges (-2) and one incoming edge from the left (1) and one incoming edge from the right (1).) We are interested in the eigenvalues and eigenvectors of these $M^{-1}K$ matrices.

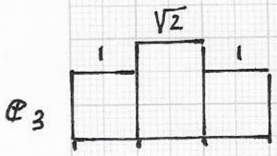
■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions: (WolframAlpha can be used for these examples.)

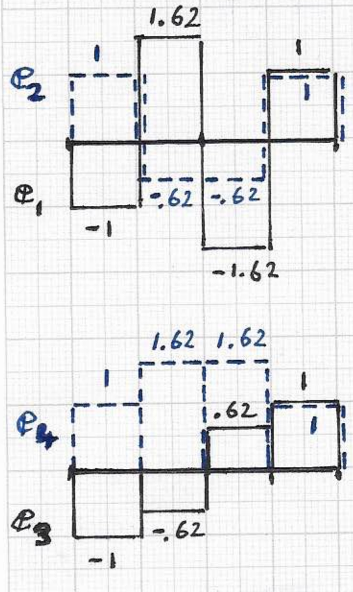


Eigenfunctions e_1, e_2 for two lixels, $m_1 = m_2 = 1$.





Eigenfunctions for three lixels, $m_1 = m_2 = m_3 = 1$; e_1, e_2, e_3.



Eigenfunctions e_1, e_2, e_3, e_4 for four lixels.

The eigenvalues for these $M^{-1}K$ matrices are:

$$\begin{bmatrix} \lambda_1 = -3 \\ \lambda_2 = -1 \end{bmatrix}, \quad \begin{bmatrix} \lambda_1 = -2-\sqrt{2} \\ \lambda_2 = -2 \\ \lambda_3 = -2+\sqrt{2} \end{bmatrix}, \quad \begin{bmatrix} \lambda_1 = (-5-\sqrt{5})/2 \\ \lambda_2 = (-3-\sqrt{5})/2 \\ \lambda_3 = (-5+\sqrt{5})/2 \\ \lambda_4 = (-3+\sqrt{5})/2 \end{bmatrix} \oplus .$$

$\oplus$ $(-5-\sqrt{5}) = -7.24$, $(-3-\sqrt{5}) = -5.24$

The corresponding eigenvectors/-functions are:

$$\begin{bmatrix} e_1 = (-1, 1)^T \\ e_2 = (1, 1)^T \end{bmatrix}, \quad \begin{bmatrix} e_1 = (1, -\sqrt{2}, 1)^T \\ e_2 = (-1, 0, 1)^T \\ e_3 = (1, \sqrt{2}, 1)^T \end{bmatrix}, \quad \begin{bmatrix} e_1 = (-1, (1+\sqrt{5})/2, -(1+\sqrt{5})/2, 1)^T \\ e_2 = (1, (1-\sqrt{5})/2, (1-\sqrt{5})/2, 1)^T \\ e_3 = (-1, (1-\sqrt{5})/2, (-1+\sqrt{5})/2, 1)^T \\ e_4 = (1, (1+\sqrt{5})/2, (1+\sqrt{5})/2, 1)^T \end{bmatrix} \bullet .$$

$\bullet$ $(1+\sqrt{5}) = 3.24$, $(1-\sqrt{5}) = -1.24$

As mentioned, it is possible to "adopt a purely graph-based approach" and assign the number of outgoing edges (= degree, valence) to a lixel i (with a '–' sign) and the number 1 to all lixels j that lixel i is connected to. When using this approach the three $M^{-1}K$ matrices are:

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}, \quad \begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix}, \quad \begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$

The eigenvalues are:

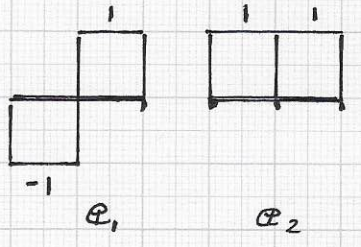
$$\begin{bmatrix} \lambda_1 = -2 \\ \lambda_2 = 0 \end{bmatrix}, \quad \begin{bmatrix} \lambda_1 = -3 \\ \lambda_2 = -1 \\ \lambda_3 = 0 \end{bmatrix}, \quad \begin{bmatrix} \lambda_1 = -2-\sqrt{2} \\ \lambda_2 = -2 \\ \lambda_3 = -2+\sqrt{2} \\ \lambda_4 = 0 \end{bmatrix}.$$

• **Note:** Last eigenvalues are all 0.

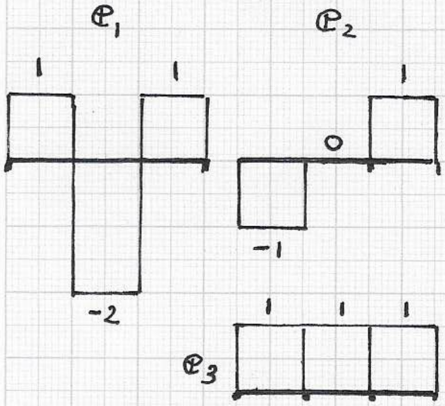
OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions: The corresponding eigenvectors/-functions are:

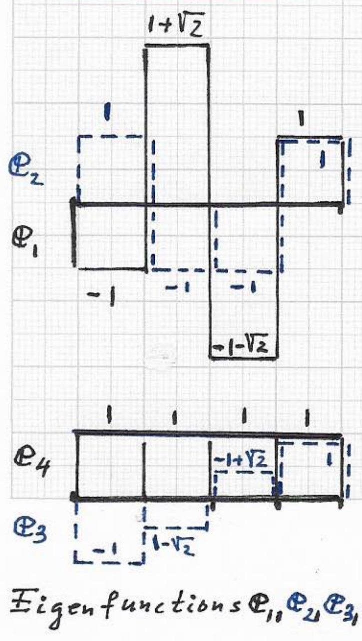
$$\begin{bmatrix} e_1 = (-1, 1)^T \\ e_2 = (1, 1)^T \end{bmatrix}, \begin{bmatrix} e_1 = (1, -2, 1)^T \\ e_2 = (-1, 0, 1)^T \\ e_3 = (1, 1, 1)^T \end{bmatrix}, \begin{bmatrix} e_1 = (-1, 1+\sqrt{2}, -1-\sqrt{2}, 1)^T \\ e_2 = (1, -1, -1, 1)^T \\ e_3 = (-1, 1-\sqrt{2}, -1+\sqrt{2}, 1)^T \\ e_4 = (1, 1, 1, 1)^T \end{bmatrix}.$$



Eigenfunctions e_1, e_2.



Eigenfunctions e_1, e_2, e_3.



Eigenfunctions e_1, e_2, e_3, e_4.

• Note: Last eigenvectors/-functions are $(1, \ldots, 1)^T$.

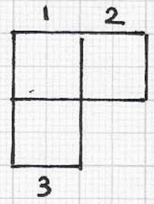
• Note: Using this graph-based approach can be viewed as a change of boundary conditions of the underlying physical model; instead of connecting mass points (lixels) on the boundary with fixed, not moving outside lixels via springs, one can let lixels on the boundary be free, not connecting them via springs to outside lixels.

• Note: Since $|\lambda_i|$ is the square of the system's i^{th} eigenfrequency (w_i^2), the eigenvalues λ_i and associated eigenvectors/-functions are indexed with respect to frequency, i.e., the last eigenvalue defines the LOWEST FREQUENCY EIGENVECTOR/-FUNCTION.

• Note: The system-specific set $\{e_i\}$ defines a SYSTEM-SPECIFIC ORTHOGONAL BASIS. !

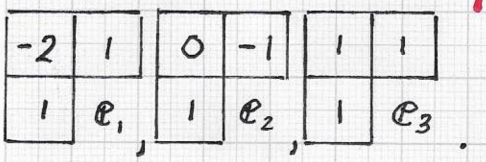
■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• *Laplacian eigenfunctions:*

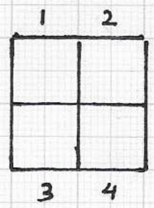


$$\begin{array}{c|c} & 1\ 2\ 3 \\ \hline \begin{matrix} \\ \\ \\ \end{matrix} & \begin{bmatrix} -2 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} \end{array}$$

$$\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3) = (-3, -1, 0)$$

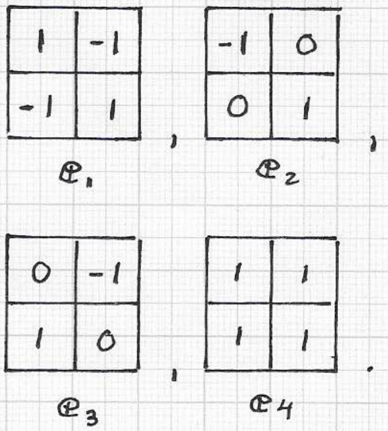


Example of 3 pixels in 2D space, all masses and edge lengths being one. Shown are the indexing; the 3×3 $M^{-1}K$ matrix; the 3 eigenvalues λ_i and -vectors e_i.



$$\begin{array}{c} 1\ \ 2\ \ 3\ \ 4 \\ \begin{bmatrix} -2 & 1 & 1 & 0 \\ 1 & -2 & 0 & 1 \\ 1 & 0 & -2 & 1 \\ 0 & 1 & 1 & -2 \end{bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} \end{array}$$

$$\boldsymbol{\lambda} = (-4, -2, -2, 0)$$



Example of 4 pixels. The indexing scheme, $M^{-1}K$ matrix, eigenvalues and -vectors are shown.

Using the described eigenvalue- and eigenfunction approach for the purpose of 3D segment characterization, analysis and classification is a viable idea for the following reasons:

1. The eigenvectors/eigenfunctions e_i define a segment-specific orthogonal basis. A segment consisting of N voxels is a piecewise constant function (mass function); the orthogonal basis has N eigenfunctions.
2. When using a single index j to index an N-voxel segment, one can view the segment as an N-dimensional vector f, a function, that can be expressed in the basis $\{e_i\}_{i=1}^N$:
$$f = \sum_{i=1}^N c_i\, e_i^* .$$
('*' denotes normalization.)
3. Since $\{e_i\}$ is an orthogonal basis, coefficient c_i is the inner product $c_i = \langle f, e_i^* \rangle$.
4. The basis $\{e_i\}$ is a basis where the index i relates to "frequency," one can compute "hierarchical f-approximations" based on frequency.

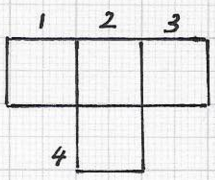
Bernd Hamann

9/23/2021

Stratovan

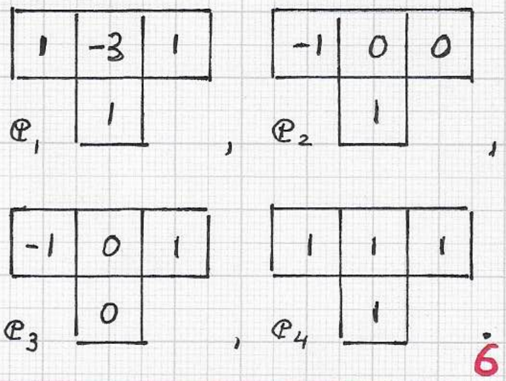
■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

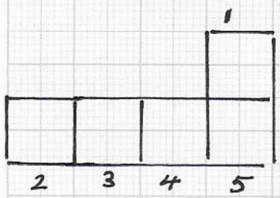
• Laplacian eigenfunctions:



$$\begin{array}{c|c} & 1\quad 2\quad 3\quad 4 \\ \hline & \begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & -3 & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} \end{array}$$

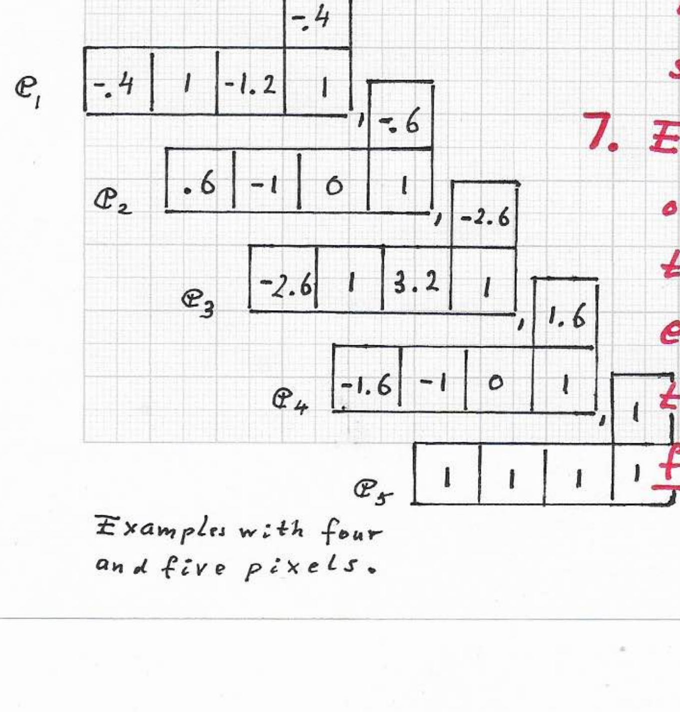
$\lambda = (-4, -1, -1, 0)$





$$\begin{array}{c|c} & 1\quad 2\quad 3\quad 4\quad 5 \\ \hline & \begin{bmatrix} -1 & 0 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \\ 1 & 0 & 0 & 1 & -2 \end{bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} \end{array}$$

$\lambda = (-3.6, -2.6, -1.4, -.4, 0)$



Examples with four and five pixels.

5. Considering the fact that certain eigenvalues correspond to high-frequency behavior, it is possible to construct – if desirable – a segment approximation f_{app} that is defined by only a subset of $\{e_i\}_{i=1}^N$: $f_{app} = \sum_J c_j e_j^*$, where J is an index set that is a subset of $I = \{1, 2, \ldots, N\}$.

6. The set of absolute eigenvalues of the segment, $\{|\lambda_i|\}_{i=1}^N$, defines a "spectrum." One can use this entire spectrum, or just a subset, to define a feature (vector) for the segment.

7. Even when a segment consists of multiple segment-components that are **NOT** connected this eigenvalue method generates the desired, meaningful eigenfrequency information for all segment-components.

~ BH