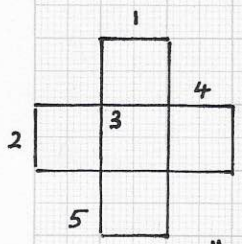
Bernd Hamann — -21- — 9/24/2021

Stratovan

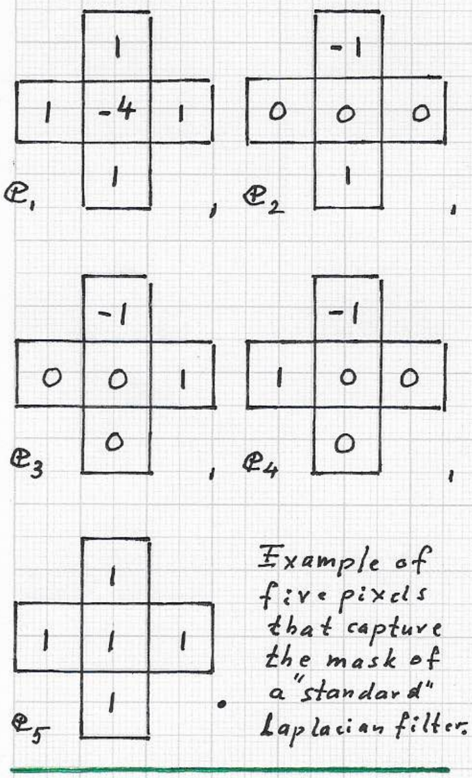
■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions:

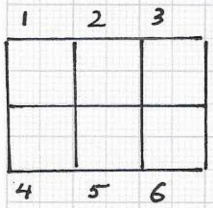


$$\begin{array}{c|c} & \begin{matrix}1 & 2 & 3 & 4 & 5\end{matrix} \\ \hline \begin{matrix}1\\2\\3\\4\\5\end{matrix} & \begin{bmatrix} -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 1 & 1 & -4 & 1 & 1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \end{bmatrix}\end{array}$$

$$\lambda = (-5, -1, -1, -1, 0)$$



Example of five pixels that capture the mask of a "standard" Laplacian filter.



$$\begin{bmatrix} -2 & 1 & 0 & 1 & 0 & 0 \\ 1 & -3 & 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 & 0 & 1 \\ 1 & 0 & 0 & -2 & 1 & 0 \\ 0 & 1 & 0 & 1 & -3 & 1 \\ 0 & 0 & 1 & 0 & 1 & -2 \end{bmatrix}$$

$$\lambda = (-5, -3, -3, -2, -1, 0)$$

Example of six pixels. Eigenvectors shown on next page.

In summary, the described eigenvalue/eigenvector approach is "based on and computes topological information (voxel connectivity, neighborhood), shape/geometrical information and mass/mass distribution information of an object in a 3D voxel data set, extracted as one segment or multiple segment-components. Most importantly, one must understand how one can use the computed eigenvalues/eigenvectors "optimally" to characterize a specific material class and/or the shape/geometry of a segment(s) of a specific material class uniquely, if possible. The additional, simple examples provide more insight into this approach.

• **Note:** The examples presented so far assume constant unit mass in each unit pixel. Thus, eigenvalues/-vectors are defined purely by the GRAPH (vertices: "cell centers", edges: connecting a vertex with its neighbors – where edges have a '–' (outgoing) or '+' sign (incoming)). When different masses are assigned to the pixels/pixel centers the data and the eigen-analysis will reflect the actual mass distribution of a segment.

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• *Laplacian eigenfunctions:* ... continuation of examples...

$$e_1 \begin{bmatrix} -1 & 2 & -1 \\ 1 & -2 & 1 \end{bmatrix}, \quad e_2 \begin{bmatrix} 1 & 0 & -1 \\ -1 & 0 & 1 \end{bmatrix}, \quad e_3 \begin{bmatrix} 0 & 1 & -1 \\ -1 & 1 & 0 \end{bmatrix}, \quad e_4 \begin{bmatrix} -1 & -1 & -1 \\ 1 & 1 & 1 \end{bmatrix},$$

$$e_5 \begin{bmatrix} -1 & 0 & 1 \\ -1 & 0 & 1 \end{bmatrix}, \quad e_6 \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Example of six pixels. One sees the progression from high-frequency behavior (e_1) to low-frequency behavior (e_6).

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \qquad \begin{bmatrix} -2 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -3 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -3 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -4 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -3 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & -2 \end{bmatrix}$$

$$\lambda = (-6, -4, -4, -3, -3, -2, -1, -1, 0)$$

(Using the "Matrix Calculator, matrixcalc.org)

$$e_1 \begin{bmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{bmatrix}, \quad e_2 \begin{bmatrix} -1 & 0 & 1 \\ 2 & 0 & -2 \\ -1 & 0 & 1 \end{bmatrix}, \quad e_3 \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix},$$

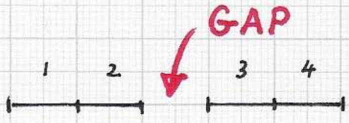
$$e_4 \begin{bmatrix} 1 & 0 & 1 \\ -1 & -2 & -1 \\ 1 & 0 & 1 \end{bmatrix}, \quad e_5 \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad e_6 \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix},$$

$$e_7 \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix}, \quad e_8 \begin{bmatrix} -1 & 0 & 1 \\ -1 & 0 & 1 \\ -1 & 0 & 1 \end{bmatrix}, \quad e_9 \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

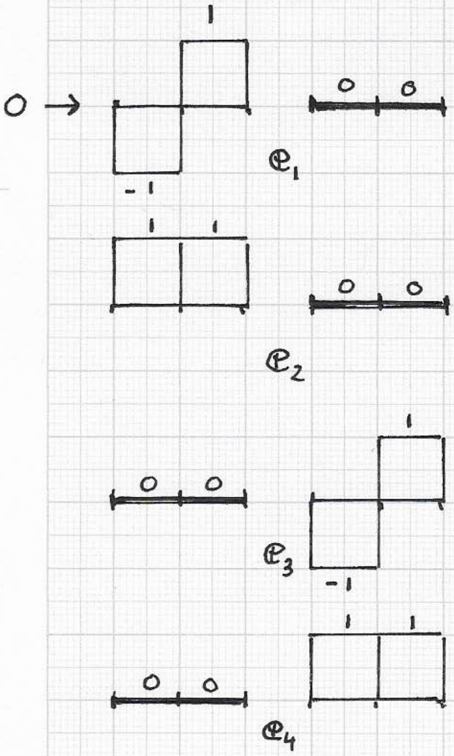
Example of nine pixels. Since three of the eigenvalues have multiplicity 2 ($\lambda_2 = \lambda_3$, $\lambda_4 = \lambda_5$ and $\lambda_7 = \lambda_8$), the pairs of eigenvectors/eigenfunctions associated with these multiple eigenvalues have the same eigenfrequencies (ω).

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions:



$$\begin{array}{c|c} & \begin{matrix}1 & 2 & 3 & 4\end{matrix} \\ \begin{matrix}1\\2\\3\\4\end{matrix} & \begin{bmatrix}-1 & 1 & 0 & 0\\ 1 & -1 & 0 & 0\\ 0 & 0 & -1 & 1\\ 0 & 0 & 1 & -1\end{bmatrix}\end{array} \qquad \begin{matrix}\lambda_1 = -2\\ \lambda_2 = -2\\ \lambda_3 = 0\\ \lambda_4 = 0\end{matrix}$$



Example of simple 1D segment fragmented into two segment-components. When using all four lixels for a single eigenanalysis, the $M^{-1}K$ matrix has two "blocks" that result from the "gap" between the two components. As a consequence, the computed eigenvalues/-vectors characterize the components as "segments in their own right."

We also should explore the behavior of this approach for simple examples where

(i) a segment consists of multiple segment-components, "not connected," and

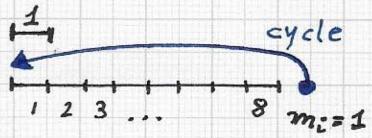
(ii) a segment with different masses associated with its pixels/voxels.

When analyzing a segment consisting of multiple segment-components, one can either perform an eigenanalysis for all components "independently" or for the entire segment with all its components in one step (using just a single enumeration of ALL the segment's pixels/voxels). The presented 1D example (left) treats a segment with two unconnected components as ONE datum to be analyzed, using just one index for the segments lixels with one gap between them. The resulting eigenanalysis yields, qualitatively, the eigenvalues/-vectors in a "concatenated form" that would also result independently when analyzing the two components separately.

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions: The examples presented so far do not involve mass values that are different for different lixels/pixels; further, the single global mass value assigned to all lixels/pixels is $m = 1$. This setting greatly simplifies the eigen-analysis of the matrix $M^{-1}K$: the analysis essentially becomes a pure "graph analysis" where only vertices and connectivity (outgoing and incoming edges) matter – while actual geometrical and physical (mass) properties are not involved. (Figures on this page show periodic cases.) We now turn to examples with varying mass, since we must characterize 3D voxel segment with varying mass properties.

PERIODIC, CYCLIC END CONDITIONS:

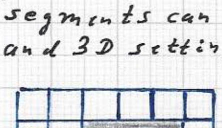


$$
\begin{array}{c|c}
 & \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array} &
\begin{bmatrix}
-2 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & -2
\end{bmatrix}
\end{array}
$$

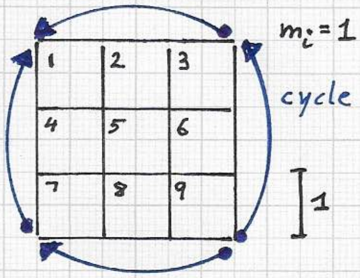
$$\lambda = (-4, -2-\sqrt{2}, -2-\sqrt{2}, -2, -2, -2+\sqrt{2}, -2+\sqrt{2}, 0)$$

$$
\begin{array}{c}
\begin{array}{cccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{array} \\ \hline
\begin{bmatrix}
-1 & -\sqrt{2} & -1 & 0 & -1 & \sqrt{2} & -1 & 1 \\
1 & 1 & \sqrt{2} & -1 & 0 & 1 & -\sqrt{2} & 1 \\
-1 & 0 & -1 & 0 & 1 & 0 & -1 & 1 \\
1 & -1 & 0 & 1 & 0 & -1 & 0 & 1 \\
-1 & \sqrt{2} & 1 & 0 & -1 & -\sqrt{2} & 1 & 1 \\
1 & -1 & -\sqrt{2} & -1 & 0 & -1 & \sqrt{2} & 1 \\
-1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\
1 & 1 & 0 & 1 & 0 & 1 & 0 & 1
\end{bmatrix}
\end{array}
$$

Example of an 8-lixel periodic set-up. Periodic segments can arise in 2D and 3D settings as well.



Periodic "RING" or "TORUS" segments can arise in 2D/3D scans.



$$
\begin{array}{c|c}
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ \\ 9 \end{array} &
\begin{bmatrix}
-4 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\
1 & -4 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\
1 & 1 & -4 & 0 & 0 & 1 & 0 & 0 & 1 \\
1 & 0 & 0 & -4 & 1 & 1 & 1 & 0 & 0 \\
 & & & & \ddots & & & & \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & -4
\end{bmatrix}
\end{array}
$$

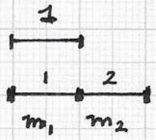
$$\lambda = (-6, -6, -6, -6, -3, -3, -3, -3, 0)$$

$$
\begin{array}{c}
\begin{array}{ccccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 & e_9 \end{array} \\ \hline
\begin{bmatrix}
1 & 1 & 1 & 1 & -1 & 0 & -1 & 0 & 1 \\
-1 & 0 & -1 & 0 & -1 & -1 & 0 & 0 & 1 \\
0 & -1 & 0 & -1 & -1 & -1 & -1 & 1 & 1 \\
-1 & -1 & 0 & 0 & 1 & 1 & 0 & -1 & 1 \\
1 & 0 & 0 & 0 & 1 & 0 & 1 & -1 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & -1 & -1 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1
\end{bmatrix}
\end{array}
$$

Example of a 9-pixel periodic set-up; periodicity is used in both dimensions.

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

•Laplacian eigenfunctions: To obtain insight into the influence of varying masses on an eigenanalysis we determine eigenvalues/-vectors for simple 1D (lixel) and 2D (pixel) examples. All examples assume that lixels are of length 1 and pixels have edge length 1; further all spring constants are of value 1. The first example considers just two masses m_1 and m_2. The general solution of the eigenanalysis problem is provided in symbolic form (left, top). The following table considers some interesting mass values:



$k_1 = k_2 = 1$

$$M^{-1}K = \begin{array}{c|cc} & 1 & 2 \\ \hline 1 & -1/m_1 & 1/m_1 \\ 2 & 1/m_2 & -1/m_2 \end{array}$$

$$\begin{bmatrix} \lambda_1 = -\frac{m_1+m_2}{m_1 m_2} \\ \lambda_2 = 0 \end{bmatrix}, \begin{bmatrix} e_1 = \left(-\frac{m_2}{m_1}, 1\right)^T \\ e_2 = (1,1)^T \end{bmatrix}.$$

•INFLUENCE OF m_1 AND m_2 ON λ_1 AND e_1:

EIGENVALUE SENSITIVITY

1) Equal masses $\Rightarrow e_1 = e_2$.
2) Equal masses, large (small) value $\Rightarrow$ small (large) $|\lambda_1|$ value $\Rightarrow$ Low (high) frequency $\sqrt{|\lambda_1|}$.
3) $m_1 \to \infty$, $m_2 = const\ (=1)$ $\Rightarrow \lambda_1 \to -1$
4) $m_1 \to 0$, $m_2 = const$ $\Rightarrow \lambda_1 \to -\infty$
5) $\lambda_1 = -\frac{m_1+m_2}{m_1 m_2} = -\left(\frac{1}{m_1}+\frac{1}{m_2}\right)$ $\Rightarrow -\lambda_1$ is the sum of the reciprocal values of m_1, m_2.

3)* $\lim\limits_{m_1\to\infty} (\lambda_1) = \lim\limits_{m_1\to\infty}\left(-\left(\frac{1}{m_1}+\frac{1}{m_2}\right)\right) = -\frac{1}{m_2}$.

m_1	m_2	λ_1	λ_2	e_1	e_2
1	1	-2	0	$(-1,1)^T$	$(1,1)^T$
100	1	$-\frac{101}{100}$	0	$(-\frac{1}{100},1)^T$	$(1,1)^T$
100	100	$-\frac{1}{50}$	0	$(-1,1)^T$	$(1,1)^T$
.001	.001	-200	0	$(-1,1)^T$	$(1,1)^T$
2	1	$-\frac{3}{2}$	0	$(-\frac{1}{2},1)^T$	$(1,1)^T$
4	1	$-\frac{5}{4}$	0	$(-\frac{1}{4},1)^T$	$(1,1)^T$
10^6	1	$-\frac{1+10^6}{10^6}$	0	$(-\frac{1}{10^6},1)^T$	$(1,1)^T$
10^{-6}	1	$-(1+10^6)$	0	$(-10^6,1)^T$	$(1,1)^T$

Note: $|\lambda_i| = \omega_i^2$ (ω_i: i^{th} eigenfrequency)

Since the values of λ_2 and e_2 are constants ($\lambda_2 = 0$, $e_2 = (1,1)^T$), we are interested in the influence of m_1 and m_2 values on λ_1 and e_1 values.