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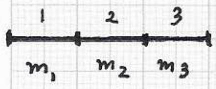
■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions: Next, we consider a 3-lixel example, using masses m_1, m_2 and m_3. Since the symbolic closed-form solution for this setting is already very complex, we consider a few specific numerical cases. Since $\lambda_3 = 0$ and $\mathbb{e}_3 = (1,1,1)^T$ for all mass values, we do not include λ_3 and $\mathbb{e}_3$ in the table.



$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} -\frac{1}{m_1} & \frac{1}{m_1} & 0 \\ \frac{1}{m_2} & -\frac{2}{m_2} & \frac{1}{m_2} \\ 0 & \frac{1}{m_3} & -\frac{1}{m_3} \end{bmatrix} \end{array}$$

$\lambda_1 = \ldots$, $\lambda_2 = \ldots$, $\lambda_3 = 0$

$\mathbb{e}_1 = \ldots$, $\mathbb{e}_2 = \ldots$, $\mathbb{e}_3 = (1,1,1)^T$

m_1	m_2	m_3	λ_1	λ_2	$\mathbb{e}_1^T$	$\mathbb{e}_2^T$	
1	1	1	−3	−2	(1, −2, 1)	(−1, 0, 1)	×
4	1	1	−2.7	−.6	(.2, −1.7, 1)	(−.4, .4, 1)	
4	4	1	−1.3	−.4	(.1, −.3, 1)	(−.8, .6, 1)	
4	4	4	−3/4	−1/4	(1, −2, 1)	(−1, 0, 1)	×
1/4	1/4	1/4	−12	−4	(1, −2, 1)	(−1, 0, 1)	×
4	2	1	−7/4	−1/2	$(\frac{1}{8}, -\frac{3}{4}, 1)$	$(-\frac{1}{2}, \frac{1}{2}, 1)$	
4	1	1/4	−21/4	−1	$(\frac{1}{64}, -\frac{5}{16}, 1)$	$(-\frac{1}{4}, \frac{3}{4}, 1)$	

• **RELATION m_i, λ_i, $\mathbb{e}_i$:**

1) × $m_1 = m_2 = m_3 \Rightarrow$ same set of eigenvectors.

2) "A large (small) mass value m_i has a small (large) associated $|\lambda_i|$ value." – "Relatively larger masses have relatively lower eigen-frequencies."

3) The eigenvalues λ_i characterize both the eigenfrequency behavior of different masses as well as the ratios of eigenfrequencies resulting from mass ratios.

Different m_i values lead to different eigenvalues λ_i – BUT NOT ALWAYS DIFFERENT EIGEN-VECTORS (×). The eigenvalues λ_i define a data set's eigenfrequencies $\omega_i = \sqrt{|\lambda_i|}$; thus, the eigenvalues λ_i, or even a subset, provide a "spectral signature" of a data set.

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■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions:

Data set = Segment

= Set of unit lixels, pixels or voxels with associated mass values

= Definition of a piecewise constant "mass function" expressed in the box function basis.

This "mass function $\mathbb{f}$" can be viewed as a point/vector in a high-dimensional space; each lixel, pixel or voxel of a segment represents one dimension.

The box function basis is NOT considering or capturing frequency behavior of a segment. BUT: The described eigen-analysis of a segment produces a data set specific MULTI-FREQUENCY BASIS THAT IS ALSO AN ORTHOGONAL BASIS. This basis has normalized basis vectors/functions $\mathbb{e}_i^{norm}$, and $\mathbb{f}$ can be written as

$$\mathbb{f} = \sum_i c_i \mathbb{e}_i^{norm}.$$

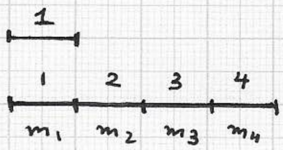
It is now possible to use a subset of the basis functions for:

- noise reduction
- data compression
- frequency-/band-specific analysis

A data set, i.e., a set of unit-edge-length lixels, pixels or voxels with associated masses, defines a unique set of eigenvalues with implied eigenvectors. We are primarily interested in "very good signatures," "very good features" of a data set, in our case a segment in a 2D image or 3D volume scan. The data set specific eigenvalues serve exactly this purpose. The eigenvectors/eigenfunctions of a data set could potentially serve a different purpose: The set of normalized and mutually orthogonal eigenfunctions $\{\mathbb{e}_i^{norm}\}$ allows one to project a given data set into this basis – A BASIS THAT CONSISTS OF **LOW- AND HIGH-FREQUENCY** BASIS FUNCTIONS – and approximate or "reconstruct" a data set with only some basis functions. The complete, lossless representation of a given (piecewise constant) mass function $\mathbb{f}$ is defined as $\mathbb{f} = \sum_i c_i \mathbb{e}_i^{norm}$, where $c_i = \langle \mathbb{f}, \mathbb{e}_i^{norm} \rangle$. For example, a lossy low-frequency-based approximation of $\mathbb{f}$ would be based on only low-frequency basis functions.

OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

- Laplacian eigenfunctions:



$k_1 = k_2 = k_3 = k_4 = 1$,

$\lambda_4 = 0$,

$M^{-1}K =$

$$\begin{bmatrix} -\frac{1}{m_1} & \frac{1}{m_1} & 0 & 0 \\ \frac{1}{m_2} & -\frac{2}{m_2} & \frac{1}{m_2} & 0 \\ 0 & \frac{1}{m_3} & -\frac{2}{m_3} & \frac{1}{m_3} \\ 0 & 0 & \frac{1}{m_4} & -\frac{1}{m_4} \end{bmatrix}.$$

Note: Of course, an orthogonal basis of a given segment is already defined by the set of lixels, pixels or voxels defining a segment, WITHOUT considering mass values (i.e., all $m_i = 1$). This "generic" basis could be used to losslessly represent any mass function f over the given grid and connectivity structure. **BUT:** The "truly data set specific representation" of a segment is the representation using its associated basis implied by grid and mass values.

- **IMPORTANT:** In general, the "λ-signatures" of two segments belonging to the same class have different eigenvalues λ_i – as a consequence of different resolutions and shapes. Despite this fact, one must still conclude that the two segments belong to the same class.

We now consider the use of the discussed eigenanalysis for eigenvalue-based SPECTRAL CHARACTERIZATION of simple lixel data. The table provides eigenvalue spectra:

m_1	m_2	m_3	m_4	λ_1	λ_2	λ_3
1	1	1	1	-3.414	-2	-0.586
2	1	1	1	-3.313	-1.758	-0.429
4	1	1	1	-3.277	-1.650	-0.324
8	1	1	1	-3.261	-1.600	-0.263
...						
4096	1	1	1	-3.247	-1.555	-0.198

$\lambda_4 = 0$

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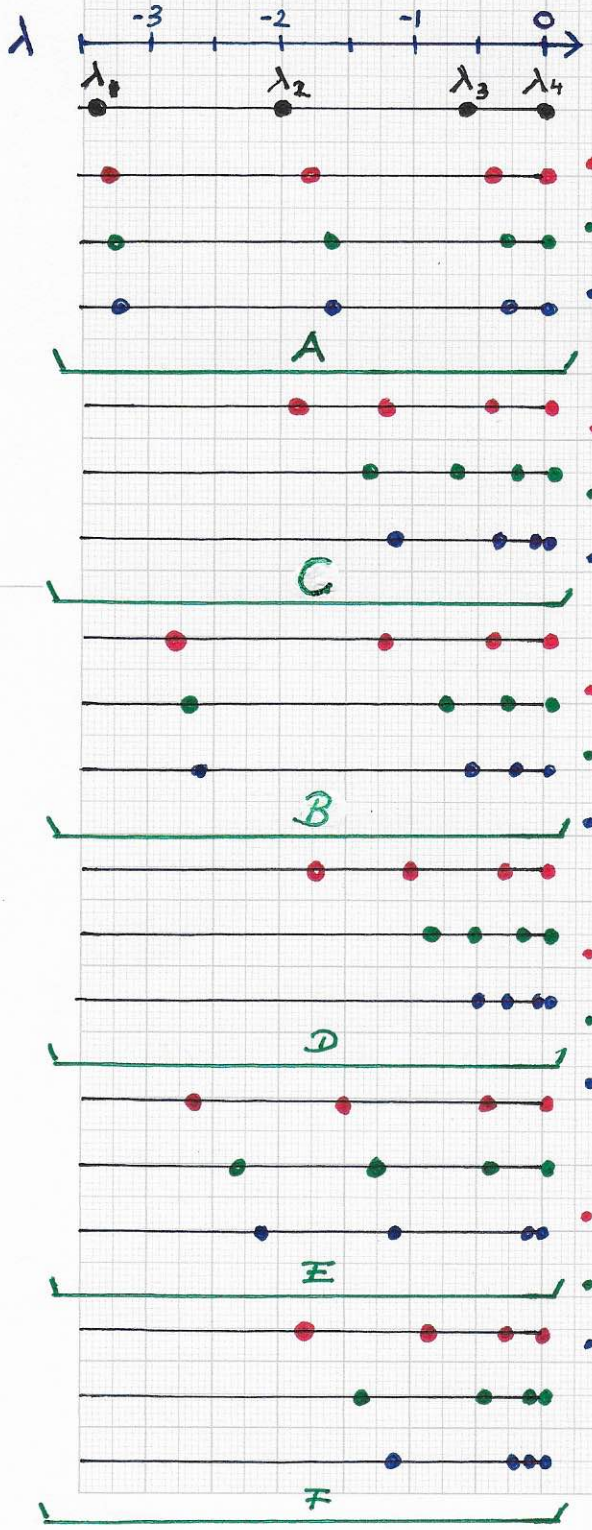
OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions: Additional spectra for four-lixel example:



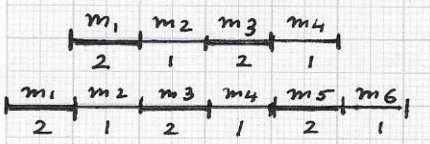
SPECTRA

m_1	m_2	m_3	m_4	λ_1	λ_2	λ_3	
2	2	1	1	-2.835	-1.238	-0.427	
4	4	1	1	-2.702	-0.732	-0.316	B
8	8	1	1	-2.656	-0.512	-0.207	
4096	4096	1	1	-2.618	-0.382	-0.0005	
2	2	2	1	-1.901	-1.223	-0.377	
4	4	4	1	-1.333	-0.670	-0.218	C
8	8	8	1	-1.143	-0.365	-0.117	
4096	4096	4096	1	-1.0002	-0.0007	-0.0002	
2	2	2	2	-1.707	-1.000	-0.293	
4	4	4	4	-0.854	-0.500	-0.146	D
8	8	8	8	-0.427	-0.250	-0.073	
4096	4096	4096	4096	-0.0008	-0.0005	-0.0001	
2	1	2	1	-2.618	-1.500	-0.382	
4	1	4	1	-2.281	-1.250	-0.219	E
8	1	8	1	-2.133	-1.125	-0.117	
4096	1	4096	1	-2.0002	-1.0002	-0.0002	
4	3	2	1	-1.789	-0.855	-0.272	
10	7	4	1	-1.389	-0.427	-0.139	F
22	15	8	1	-1.142	-0.217	-0.070	
12286	8191	4096	1	-1.0002	-0.0004	-0.0001	

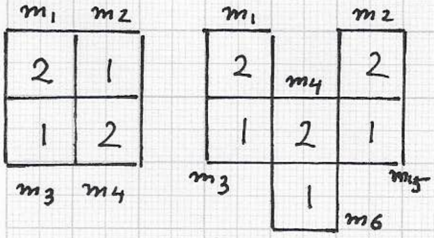
Table with "interesting" mass values; λ_i-values visualized on the λ-line (left).

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

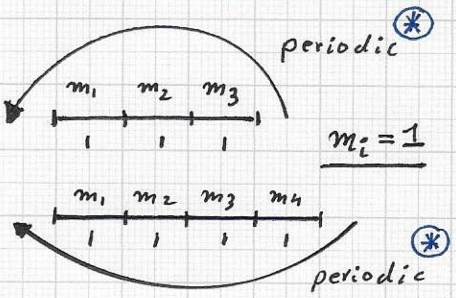
• Laplacian eigenfunctions:



Example of two lixel segments belonging to the "same material class."



Example of two pixel segments belonging to the "same material class."



Example of two "very simple objects" of the same class: m_i = constant = 1. Here, the same material is given as a 3-lixel and 4-lixel object/segment. The signatures of the two objects, based on eigenvalue analysis, must be identical. Thus, the signature must be independent of total size, total mass ($=\sum m_i$) and resolution of the segments.

The eigenvalues computed for a segment also depend on:

- total mass of segment, m_i-values
- resolution of segment (number of lixels, pixels, voxels)
- shape/geometry and connectivity of grid representing the segment

It is crucial to understand how eigenvalues depend on these parameters. Understanding this dependency should make it possible to derive a subsequent segment signature that would be (nearly) identical for two segments that belong to the same material class (see 1D and 2D examples, left).

First, we consider the case of the same material represented with a 3- and 4-lixel object (Left, bottom). The $M^{-1}K$ matrices are:

$$\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}^{*}, \quad \begin{bmatrix} -2 & 1 & 0 & 1 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 1 & 0 & 1 & -2 \end{bmatrix}^{*}.$$

$\lambda = (-3, -3, 0)$ $\qquad$ $\lambda = (-4, -2, -2, 0)$

* We use periodic end conditions for simplicity.

The resulting eigenvalue tuples λ for the 3- and 4-lixel objects have different numbers of dimensions (3 and 4) and different values. Thus, it is necessary to use mappings leading to two identical signatures.