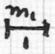
Bernd Hamann — -6- — 10/1/2021

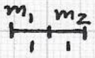
Stratovan

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

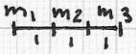
• Laplacian eigenfunctions: We consider the simple 1D case of equal, unit mass values ($m_i = 1$) for resolutions 1,…, 8. The resulting eigenvalue spectra and eigenvectors/-functions are summarized (periodic case):



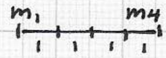
$\lambda = (0)$ or $\lambda = (-1)$



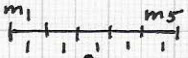
$\lambda = (-2, 0)$



$\lambda = (-3, -3, 0)$

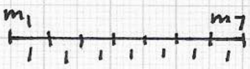


$\lambda = (-4, -2, -2, 0)$

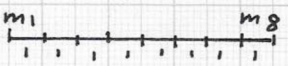


$\lambda = ((-5-\sqrt{5})/2, (-5-\sqrt{5})/2, (-5+\sqrt{5})/2, (-5+\sqrt{5})/2, 0)$

$\lambda = (-4, -3, -3, -1, -1, 0)$



$\lambda = (-3.80, -3.80, -2.45, -2.45, -.75, -.75, 0)$



$\lambda = (-4, -2-\sqrt{2}, -2-\sqrt{2}, -2, -2, -2+\sqrt{2}, -2+\sqrt{2}, 0)$

Eigenvalue tuples (top) and corresponding eigenvectors for 1D case – unit spacing and unit masses. The $M^{-1}K$ matrices use periodic/cyclic boundary conditions (right).

$M^{-1}K$	e_i^T, $i = 1, 2, \ldots$
$[0]$ or $[-1]$	(1)
$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$	(-1, 1) (1, 1)
$\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}$	(-1, 0, 1) (-1, 1, 0) (1, 1, 1)
$\begin{bmatrix} -2 & 1 & 0 & 1 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 1 & 0 & 1 & -2 \end{bmatrix}$	(-1, 1, -1, 1) (0, -1, 0, 1) (-1, 0, 1, 0) (1, 1, 1, 1)
$\begin{bmatrix} -2 & 1 & 0 & 0 & 1 \\ 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \\ 1 & 0 & 0 & 1 & -2 \end{bmatrix}$	(-1.6, 1.6, -1, 0, 1) (-1, 1.6, -1.6, 1, 0) (.6, -.6, -1, 0, 1) (-1, -.6, .6, 1, 0) (1, 1, 1, 1, 1)
$\begin{bmatrix} -2 & 1 & 0 & 0 & 0 & 1 \\ 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 \\ 1 & 0 & 0 & 0 & 1 & -2 \end{bmatrix}$	(-1, 1, -1, 1, -1, 1) (-1, 0, 1, -1, 0, 1) (-1, 1, 0, -1, 1, 0) (1, 0, -1, -1, 0, 1) (-1, -1, 0, 1, 1, 0) (1, 1, 1, 1, 1, 1)
$\begin{bmatrix} -2 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & -2 \end{bmatrix}$	(-1.80, 2.25, -2.25, 1.80, -1, 0, 1) (-1, 1.80, -2.25, 2.25, -1.80, 1, 0) (-.45, -.80, .80, .45, -1, 0, 1) (-1, .45, .80, -.80, -.45, 1, 0) (1.25, .55, -.55, -1.25, -1, 0, 1) (-1, -1.25, -.55, .55, 1.25, 1, 0) (1, 1, 1, 1, 1, 1, 1)
…	…

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions:

$(M^{-1}K)\,\mathbb{e} = \lambda \mathbb{e}$,

$(M^{-1}K - \lambda I)\,\mathbb{e} = \mathbb{0}$,

$\det(M^{-1}K - \lambda I) = 0$,

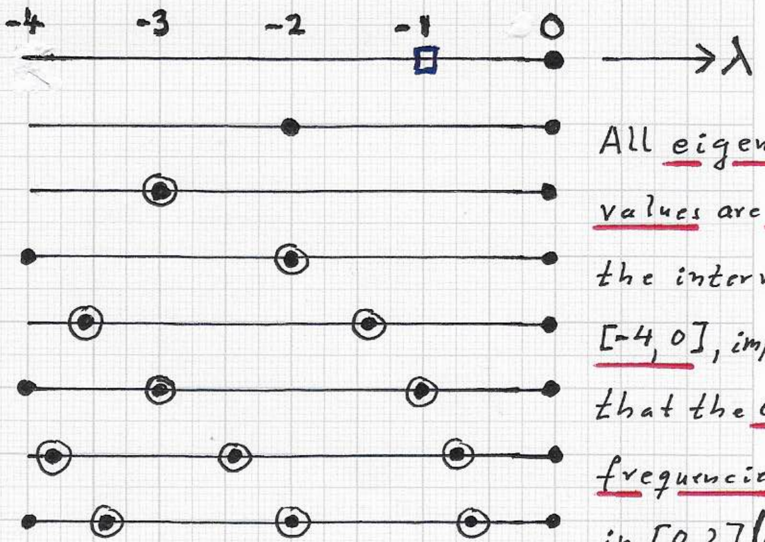
$\mathrm{cpol}(\lambda) = 0$.

$\dddot{M}^{-1}K$	$\mathbb{e}_i^T$
$\begin{bmatrix} -2 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & -2 \end{bmatrix}$	$(-1, 1, -1, 1, -1, 1, -1, 1)$ $(-\sqrt{2}, 1, 0, -1, \sqrt{2}, -1, 0, 1)$ $(-1, \sqrt{2}, -1, 0, 1, -\sqrt{2}, 1, 0)$ $(0, -1, 0, 1, 0, -1, 0, 1)$ $(-1, 0, 1, 0, -1, 0, 1, 0)$ $(\sqrt{2}, 1, 0, -1, -\sqrt{2}, -1, 0, 1)$ $(-1, -\sqrt{2}, -1, 0, 1, \sqrt{2}, 1, 0)$ $(1, 1, 1, 1, 1, 1, 1, 1)$

The characteristic polynomial $\mathrm{cpol}(\lambda)$ associated with an $M^{-1}K$ matrix defines the eigenvalue spectrum of an n-lixel data set.

Eigenvalue spectra of symmetric, real, tri-diagonal matrices are a special spectral class. These spectra can be expressed in closed form.

The λ_i-spectra for these eight simple 1D cases can be visualized on the λ-line:



All eigenvalues are in the interval $[-4, 0]$, implying that the eigenfrequencies are in $[0, 2]$ $(\omega = \sqrt{|\lambda|})$.

Eigenvalues λ_i and λ_{i+1} can have the same value, i.e., this (eigen)value has multiplicity 2; this is shown on the λ-line via the '⊙' symbol. Eigenvalues have multiplicity 1 or 2 – which is a consequence of the odd or even degree of the characteristic polynomial $\mathrm{cpol}(\lambda)$ associated with a case.

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• **Laplacian eigenfunctions:** We also consider the 1D, non-periodic ("open") case:

$\lambda = (-2),$

$= (-3, -1),$

$= (-2-\sqrt{2}, -2, -2+\sqrt{2}),$

$= (-3.62, -2.62, -1.38, -.38),$

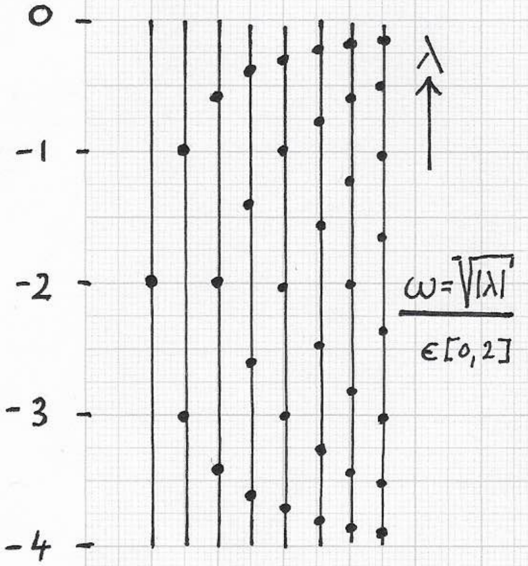
$= (-2-\sqrt{3}, -3, -2, -1, -2+\sqrt{3}),$

$= (-3.8, -3.25, -2.45, -1.55, -.75, -.198),$

$= (-3.85, -2-\sqrt{2}, -2.77, -2,$
$\quad -1.23, -2+\sqrt{2}, -.15),$

$= (-3.88, -3.52, -3, -2.35,$
$\quad -1.65, -1, -.47, -.12).$

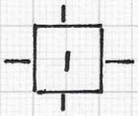
$M^{-1}K$	e_i^T, $i = 1, \dots, 8$
$\begin{bmatrix} -2 \end{bmatrix}$	(1)
$\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$	$(-1, 1)$ $(1, 1)$
$\begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix}$	$(1, -\sqrt{2}, 1)$ $(-1, 0, 1)$ $(1, \sqrt{2}, 1)$
$\begin{bmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}$	$(-1, 1.62, -1.62, 1)$ $(1, -.62, -.62, 1)$ $(-1, -.62, .62, 1)$ $(1, 1.62, 1.62, 1)$
$\begin{bmatrix} -2 & 1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 & -2 \end{bmatrix}$	$(1, -\sqrt{3}, 2, -\sqrt{3}, 1)$ $(-1, 1, 0, -1, 1)$ $(1, 0, -1, 0, 1)$ $(-1, -1, 0, 1, 1)$ $(1, \sqrt{3}, 2, \sqrt{3}, 1)$
$\begin{bmatrix} -2 & 1 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 1 & -2 \end{bmatrix}$	$(-1, 1.80, -2.25, 2.25, -1.80, 1)$ $(1, -1.25, .55, .55, -1.25, 1)$ $(-1, .45, .80, -.80, -.45, 1)$ $(1, .45, -.80, -.80, .45, 1)$ $(-1, -1.25, -.55, .55, 1.25, 1)$ $(1, 1.80, 2.25, 2.25, 1.80, 1)$
$\begin{bmatrix} -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 \end{bmatrix}$	$(1, -1.8, 2.4, -2.6, 2.4, -1.8, 1)$ $(-1, \sqrt{2}, -1, 0, 1, -\sqrt{2}, 1)$ $(1, -.8, -.4, 1.1, -.4, -.8, 1)$ $(-1, 0, 1, 0, -1, 0, 1)$ $(1, .8, -.4, -1.1, -.4, .8, 1)$ $(-1, -\sqrt{2}, -1, 0, 1, \sqrt{2}, 1)$ $(1, 1.8, 2.4, 2.6, 2.4, 1.8, 1)$
$\begin{bmatrix} -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 \end{bmatrix}$	$(-1, 1.9, -2.5, 2.9, -2.9, 2.5, -1.9, 1)$ $(1, -1.5, 1.3, -.5, -.5, 1.3, -1.5, 1)$ $(-1, 1, 0, -1, 1, 0, -1, 1)$ $(1, -.3, -.9, .7, .7, -.9, -.3, 1)$ $(-1, -.3, .9, .7, -.7, -.9, .3, 1)$ $(1, 1, 0, -1, -1, 0, 1, 1)$ $(-1, -1.5, -1.3, -.5, .5, 1.3, 1.5, 1)$ $(1, 1.9, 2.5, 2.9, 2.9, 2.5, 1.9, 1)$



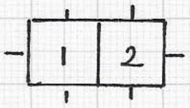
Eigenvalue spectra for resolutions 1, ..., 8 (from left to right). Spectra show a "pattern."

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

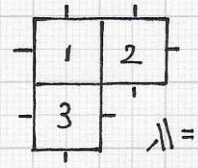
• *Laplacian eigenfunctions:* We now present a few 2D examples of grids consisting of unit squares with unit mass values ($m_i = 1$). For simplicity, we always assign the "out-degree" −4 to a grid square when defining its associated row in the $M^{-1}K$ matrix. In the images (left) we show a square's index in its interior.



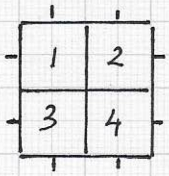
$\lambda = (-4)$



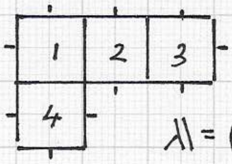
$\lambda = (-5, -3)$



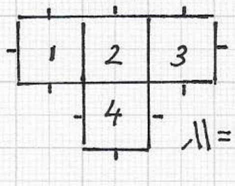
$\lambda = (-4-\sqrt{2}, -4, -4+\sqrt{2})$



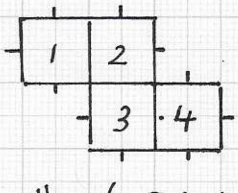
$\lambda = (-6, -4, -4, -2)$



$\lambda = (-5.6, -4.6, -3.4, -2.4)$



$\lambda = (-4-\sqrt{3}, -4, -4, -4+\sqrt{3})$



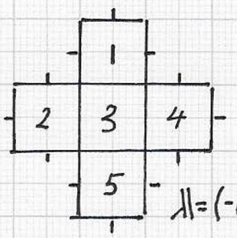
$\lambda = (-5.6, -4.6, -3.4, -2.4)$

Simple 1-, 2-, 3- and 4-pixel, "rectilinear unit square," grids, with different connectivity for 4 pixels.

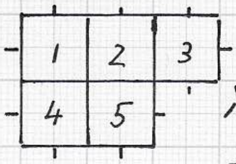
$M^{-1}K$	e_i^T, $i = 1, 2, \ldots$
$\begin{bmatrix} -4 \end{bmatrix}$	(1)
$\begin{bmatrix} -4 & 1 \\ 1 & -4 \end{bmatrix}$	$(-1, 1)$ $(1, 1)$
$\begin{bmatrix} -4 & 1 & 1 \\ 1 & -4 & 0 \\ 1 & 0 & -4 \end{bmatrix}$	$(-\sqrt{2}, 1, 1)$ $(0, -1, 1)$ $(\sqrt{2}, 1, 1)$
$\begin{bmatrix} -4 & 1 & 1 & 0 \\ 1 & -4 & 0 & 1 \\ 1 & 0 & -4 & 1 \\ 0 & 1 & 1 & -4 \end{bmatrix}$	$(1, -1, -1, 1)$ $(-1, 0, 0, 1)$ $(0, -1, 1, 0)$ $(1, 1, 1, 1)$
$\begin{bmatrix} -4 & 1 & 0 & 1 \\ 1 & -4 & 1 & 0 \\ 0 & 1 & -4 & 0 \\ 1 & 0 & 0 & -4 \end{bmatrix}$	$(-1.6, 1.6, -1, 1)$ $(-.6, -.6, 1, 1)$ $(.6, -.6, -1, 1)$ $(1.6, 1.6, 1, 1)$
$\begin{bmatrix} -4 & 1 & 0 & 0 \\ 1 & -4 & 1 & 1 \\ 0 & 1 & -4 & 0 \\ 0 & 1 & 0 & -4 \end{bmatrix}$	$(1, -\sqrt{3}, 1, 1)$ $(-1, 0, 0, 1)$ $(-1, 0, 1, 0)$ $(1, \sqrt{3}, 1, 1)$
$\begin{bmatrix} -4 & 1 & 0 & 0 \\ 1 & -4 & 1 & 0 \\ 0 & 1 & -4 & 1 \\ 0 & 0 & 1 & -4 \end{bmatrix}$	$(-1, 1.6, -1.6, 1)$ $(1, -.6, -.6, 1)$ $(-1, -.6, .6, 1)$ $(1, 1.6, 1.6, 1)$

OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

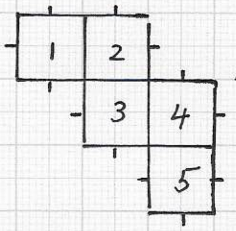
• Laplacian eigenfunctions:



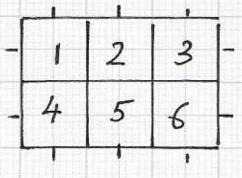
$\lambda = (-6, -4, -4, -4, -2)$



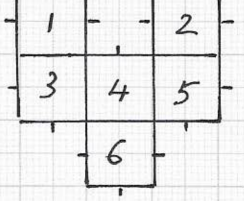
$\lambda = (-6.1, -4.7, -4, -3.3, -1.9)$



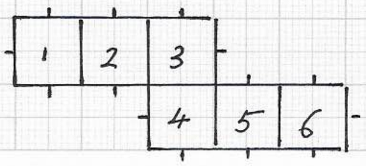
$\lambda = (-4-\sqrt{3}, -5, -4, -3, -4+\sqrt{3})$



$\lambda = (-6.4, -5, -4.4, -3.6, -3, -1.6)$



$\lambda = (-5.9, -5, -4.5, -3.5, -3, -2.1)$



$\lambda = (-5.8, -5.2, -4.4, -3.6, -2.8, -2.2)$

$M^{-1}K$	e_i^T, $i = 1, 2, \ldots$
$\begin{bmatrix} -4 & 0 & 1 & 0 & 0 \\ 0 & -4 & 1 & 0 & 0 \\ 1 & 1 & -4 & 1 & 1 \\ 0 & 0 & 1 & -4 & 0 \\ 0 & 0 & 1 & 0 & -4 \end{bmatrix}$	$(1, 1, -2, 1, 1)$ $(-1, 0, 0, 0, 1)$ $(-1, 0, 0, 1, 0)$ $(-1, 1, 0, 0, 0)$ $(1, 1, 2, 1, 1)$
$\begin{bmatrix} -4 & 1 & 0 & 1 & 0 \\ 1 & -4 & 1 & 0 & 1 \\ 0 & 1 & -4 & 0 & 0 \\ 1 & 0 & 0 & -4 & 1 \\ 0 & 1 & 0 & 1 & -4 \end{bmatrix}$	$(1, -1.2, .6, -.9, 1)$ $(1, 2.4, -3.6, -3.0, 1)$ $(-1, 0, 0, 0, 1)$ $(1, -2.4, -3.6, 3.0, 1)$ $(1, 1.2, .6, .9, 1)$
$\begin{bmatrix} -4 & 1 & 0 & 0 & 0 \\ 1 & -4 & 1 & 0 & 0 \\ 0 & 1 & -4 & 1 & 0 \\ 0 & 0 & 1 & -4 & 1 \\ 0 & 0 & 0 & 1 & -4 \end{bmatrix}$	$(1, -\sqrt{3}, 2, -\sqrt{3}, 1)$ $(-1, 1, 0, -1, 1)$ $(1, 0, -1, 0, 1)$ $(-1, -1, 0, 1, 1)$ $(1, \sqrt{3}, 2, \sqrt{3}, 1)$
$\begin{bmatrix} -4 & 1 & 0 & 1 & 0 & 0 \\ 1 & -4 & 1 & 0 & 1 & 0 \\ 0 & 1 & -4 & 0 & 0 & 1 \\ 1 & 0 & 0 & -4 & 1 & 0 \\ 0 & 1 & 0 & 1 & -4 & 1 \\ 0 & 0 & 1 & 0 & 1 & -4 \end{bmatrix}$	$(-1, \sqrt{2}, -1, 1, -\sqrt{2}, 1)$ $(1, 0, -1, -1, 0, 1)$ $(1, -\sqrt{2}, 1, 1, -\sqrt{2}, 1)$ $(-1, -\sqrt{2}, -1, 1, \sqrt{2}, 1)$ $(-1, 0, 1, -1, 0, 1)$ $(1, \sqrt{2}, 1, 1, \sqrt{2}, 1)$
$\begin{bmatrix} -4 & 0 & 1 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 & 1 & 0 \\ 1 & 0 & -4 & 1 & 0 & 0 \\ 0 & 0 & 1 & -4 & 1 & 1 \\ 0 & 1 & 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 1 & 0 & -4 \end{bmatrix}$	$(-.7, -.7, 1.4, -1.9, 1.4, 1)$ $(1, -1, -1, 0, 1, 0)$ $(.7, .7, -.4, -.5, -.4, 1)$ $(-.7, -.7, -.4, .5, -.4, 1)$ $(-1, 1, -1, 0, 1, 0)$ $(.7, .7, 1.4, 1.9, 1.4, 1)$
$\begin{bmatrix} -4 & 1 & 0 & 0 & 0 & 0 \\ 1 & -4 & 1 & 0 & 0 & 0 \\ 0 & 1 & -4 & 1 & 0 & 0 \\ 0 & 0 & 1 & -4 & 1 & 0 \\ 0 & 0 & 0 & 1 & -4 & 1 \\ 0 & 0 & 0 & 0 & 1 & -4 \end{bmatrix}$	$(-1, 1.8, -2.2, 2.2, -1.8, 1)$ $(1, -1.2, .6, .6, -1.2, 1)$ $(-1, .4, .8, -.8, -.4, 1)$ $(1, .4, -.8, -.8, .4, 1)$ $(-1, -1.2, -.6, .6, 1.2, 1)$ $(1, 1.8, 2.2, 2.2, 1.8, 1)$

Examples: 5-pixel and 6-pixel configurations with resulting $M^{-1}K$ matrices and their eigenvalues and eigenvectors.