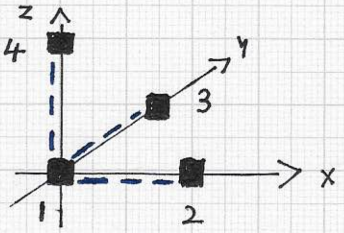
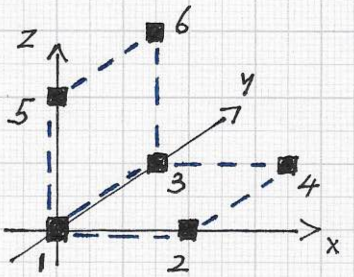
■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

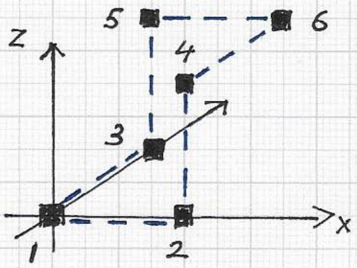
• Laplacian applications:



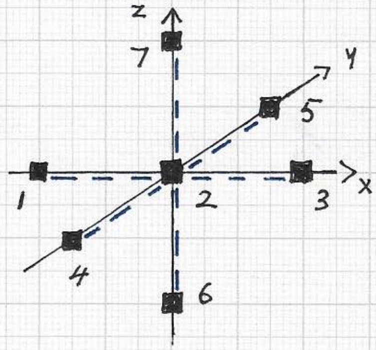
$\lambda = (-6-\sqrt{3}, -6, -6, -6+\sqrt{3})$



$\lambda = (-7-\sqrt{2}, -7, -5-\sqrt{2}, -7+\sqrt{2}, -5, -5+\sqrt{2})$



$\lambda = (-8, -7, -7, -5, -5, -4)$



$\lambda = (-6-\sqrt{6}, -6, -6, -6, -6, -6, -6+\sqrt{6})$

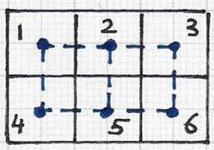
Concerning the 3D case, we only consider a small number of very simple cases. The 3D grids are rectilinear grids consisting of unit cubes with unit mass values ($m_i = 1$). When defining row i in the $M^{-1}K$ matrix, for grid cube i, we always use the "out-degree" -6.

$M^{-1}K$	e_i^T, $i = 1, 2, \ldots$
$\begin{bmatrix} -6 & 1 & 1 & 1 \\ 1 & -6 & 0 & 0 \\ 1 & 0 & -6 & 0 \\ 1 & 0 & 0 & -6 \end{bmatrix}$	$(-\sqrt{3}, 1, 1, 1)$ $(0, -1, 0, 1)$ $(0, -1, 1, 0)$ $(\sqrt{3}, 1, 1, 1)$
$\begin{bmatrix} -6 & 1 & 1 & 0 & 1 & 0 \\ 1 & -6 & 0 & 1 & 0 & 0 \\ 1 & 0 & -6 & 1 & 0 & 1 \\ 0 & 1 & 1 & -6 & 0 & 0 \\ 1 & 0 & 0 & 0 & -6 & 1 \\ 0 & 0 & 1 & 0 & 1 & -6 \end{bmatrix}$	$(\sqrt{2}, -1, -\sqrt{2}, 1, -1, 1)$ $(0, 1, 0, -1, -1, 1)$ $(-\sqrt{2}, 1, -\sqrt{2}, 1, 1, 1)$ $(-\sqrt{2}, -1, \sqrt{2}, 1, -1, 1)$ $(0, -1, 0, -1, 1, 1)$ $(\sqrt{2}, 1, \sqrt{2}, 1, 1, 1)$
$\begin{bmatrix} -6 & 1 & 1 & 0 & 0 & 0 \\ 1 & -6 & 0 & 1 & 0 & 0 \\ 1 & 0 & -6 & 0 & 1 & 0 \\ 0 & 1 & 0 & -6 & 0 & 1 \\ 0 & 0 & 1 & 0 & -6 & 1 \\ 0 & 0 & 0 & 1 & 1 & -6 \end{bmatrix}$	$(-1, 1, 1, -1, -1, 1)$ $(1, 0, -1, -1, 0, 1)$ $(0, 1, -1, -1, 1, 0)$ $(-1, 0, -1, 1, 0, 1)$ $(0, -1, 1, -1, 1, 0)$ $(1, 1, 1, 1, 1, 1)$
$\begin{bmatrix} -6 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -6 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & -6 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -6 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -6 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -6 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & -6 \end{bmatrix}$	$(1, -\sqrt{6}, 1, 1, 1, 1, 1)$ $(-1, 0, 0, 0, 0, 0, 1)$ $(-1, 0, 0, 0, 0, 1, 0)$ $(-1, 0, 0, 0, 1, 0, 0)$ $(-1, 0, 0, 1, 0, 0, 0)$ $(-1, 0, 1, 0, 0, 0, 0)$ $(1, \sqrt{6}, 1, 1, 1, 1, 1)$

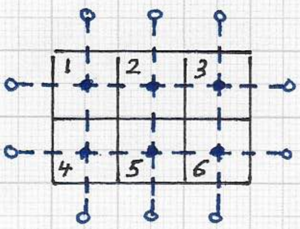
Selected configurations for 3D case. Cubes are drawn as '■', and broken lines indicate cube connectivity in the figures.

OBJECT AND MATERIAL EIGENFUNCTIONS - Cont'd.

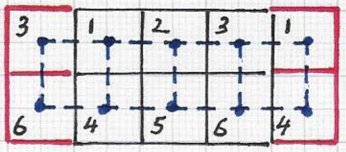
• Laplacian eigenfunctions: End conditions applied to the lixels/pixels/voxels of a segment influence the eigenvalues/-vectors resulting from the eigenanalysis of the $M^{-1}K$ matrix. We consider a six-pixel segment with unit mass values ($m_i = 1$) to gain some insight.

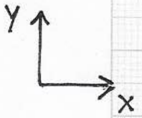


$\lambda = (-5, -3, -3, -2, -1, 0)$

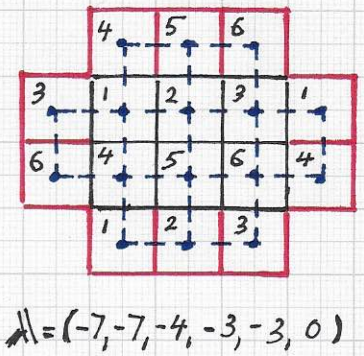


$\lambda = (-5-\sqrt{2}, -5, -3-\sqrt{2}, -5+\sqrt{2}, -3, -3+\sqrt{2})$





$\lambda = (-5, -5, -3, -3, -2, 0)$



$\lambda = (-7, -7, -4, -3, -3, 0)$

$M^{-1}K$	$\mathbf{e}_i^T,\ i = 1,\ldots,6$
$\begin{bmatrix} -2 & 1 & 0 & 1 & 0 & 0 \\ 1 & -3 & 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 & 0 & 1 \\ 1 & 0 & 0 & -2 & 1 & 0 \\ 0 & 1 & 0 & 1 & -3 & 1 \\ 0 & 0 & 1 & 0 & 1 & -2 \end{bmatrix}$	$(-1, 2, -1, 1, -2, 1)$ $(1, 0, -1, -1, 0, 1)$ $(0, 1, -1, -1, 1, 0)$ $(-1, -1, -1, 1, 1, 1)$ $(-1, 0, 1, -1, 0, 1)$ $(1, 1, 1, 1, 1, 1)$
$\begin{bmatrix} -4 & 1 & 0 & 1 & 0 & 0 \\ 1 & -4 & 1 & 0 & 1 & 0 \\ 0 & 1 & -4 & 0 & 0 & 1 \\ 1 & 0 & 0 & -4 & 1 & 0 \\ 0 & 1 & 0 & 1 & -4 & 1 \\ 0 & 0 & 1 & 0 & 1 & -4 \end{bmatrix}$	$(-1, \sqrt{2}, -1, 1, -\sqrt{2}, 1)$ $(1, 0, -1, -1, 0, 1)$ $(1, -\sqrt{2}, 1, 1, -\sqrt{2}, 1)$ $(-1, -\sqrt{2}, -1, 1, \sqrt{2}, 1)$ $(-1, 0, 1, -1, 0, 1)$ $(1, \sqrt{2}, 1, 1, \sqrt{2}, 1)$
$\begin{bmatrix} -3 & 1 & 1 & 1 & 0 & 0 \\ 1 & -3 & 1 & 0 & 1 & 0 \\ 1 & 1 & -3 & 0 & 0 & 1 \\ 1 & 0 & 0 & -3 & 1 & 1 \\ 0 & 1 & 0 & 1 & -3 & 1 \\ 0 & 0 & 1 & 1 & 1 & -3 \end{bmatrix}$	$(1, 0, -1, -1, 0, 1)$ $(1, -1, 0, -1, 1, 0)$ $(-1, 0, 1, -1, 0, 1)$ $(-1, 1, 0, -1, 1, 0)$ $(-1, -1, -1, 1, 1, 1)$ $(1, 1, 1, 1, 1, 1)$
$\begin{bmatrix} -4 & 1 & 1 & 2 & 0 & 0 \\ 1 & -4 & 1 & 0 & 2 & 0 \\ 1 & 1 & -4 & 0 & 0 & 2 \\ 2 & 0 & 0 & -4 & 1 & 1 \\ 0 & 2 & 0 & 1 & -4 & 1 \\ 0 & 0 & 2 & 1 & 1 & -4 \end{bmatrix}$	$(1, 0, -1, -1, 0, 1)$ $(1, -1, 0, -1, 1, 0)$ $(-1, -1, -1, 1, 1, 1)$ $(-1, 0, 1, -1, 0, 1)$ $(-1, 1, 0, -1, 1, 0)$ $(1, 1, 1, 1, 1, 1)$

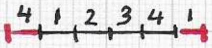
Four six-pixel examples demonstrating the effect of various end conditions (top to bottom): clamped; open; periodic in x and clamped in y; periodic in x and y.

Stratovan

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions:

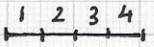
periodic



$\lambda = (-4, -2, -2, 0)$ $\quad \begin{bmatrix} -2 & 1 & 0 & 1 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 1 & 0 & 1 & -2 \end{bmatrix}$

$e_1^T = (-1, 1, -1, 1)$
$e_2^T = (0, -1, 0, 1)$
$e_3^T = (-1, 0, 1, 0)$
$e_4^T = (1, 1, 1, 1)$

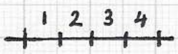
clamped



$\lambda = (-2-\sqrt{2}, -2, -2+\sqrt{2}, 0)$ $\quad \begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$

$e_1^T = (-1, 1+\sqrt{2}, -(1+\sqrt{2}), 1)$
$e_2^T = (1, -1, -1, 1)$
$e_3^T = (-1, 1-\sqrt{2}, -(1-\sqrt{2}), 1)$
$e_4^T = (1, 1, 1, 1)$

open



$\lambda = (-5-\sqrt{5}, -3-\sqrt{5}, -5+\sqrt{5}, -3+\sqrt{5})/2$ $\quad \begin{bmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}$

$e_1^T = (-1, (1+\sqrt{5})/2, -(1+\sqrt{5})/2, 1)$
$e_2^T = (1, (1-\sqrt{5})/2, (1-\sqrt{5})/2, 1)$
$e_3^T = (-1, (1-\sqrt{5})/2, -(1-\sqrt{5})/2, 1)$
$e_4^T = (1, (1+\sqrt{5})/2, (1+\sqrt{5})/2, 1)$

Eigenanalysis of 4-lixel data with three different end conditions, leading to different sets of eigenvalues and -vectors. Each set of eigenvectors e_i defines an orthonormal multi-frequency basis of the space of 4-lixel data.

(All m_i-values are one.)

The eigenanalysis examples presented for simple $M^{-1}K$ matrices demonstrate various important facts:

1) $M^{-1}K$ matrices are sparse, due to the nature of small local neighborhoods.

2) Eigenvalues can have multiplicities greater than one.

3) The eigenvectors/-functions are mutually orthogonal to each other and define an orthonormal basis.

4) THE EIGENFUNCTIONS DEFINE A MULTI-FREQUENCY ORTHONORMAL BASIS, since each eigenvalue λ has the associated (angular) frequency $\omega = \sqrt{|\lambda|}$.

5) Denoting the normalized version of an eigenfunction e by e^n, where $e^n = e / \sqrt{\langle e, e \rangle}$, makes it possible to represent any function f in the orthonormal basis $\{e_i^n\}_{i=1}^{N}$, i.e.,

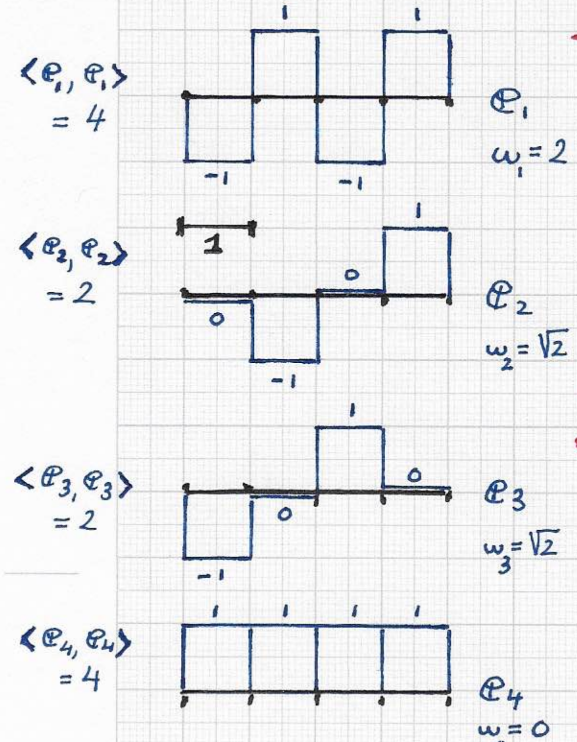
$$f = \sum_{i=1}^{N} c_i \, e_i^n, \text{ where } c_i = \langle f, e_i^n \rangle .$$

Here, a "function" is a piecewise constant function – having a constant value per lixel/pixel/voxel. A function f can thus be viewed as a point/vector in N-dimensional space.

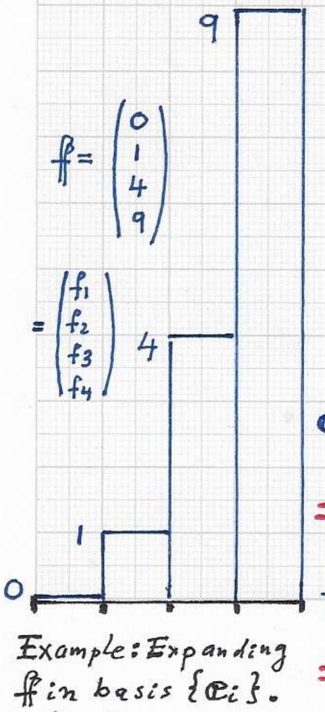
Stratovan

OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions: . . . Facts . . .



Eigenfunctions of 4-lixel data based on periodic end conditions (not normalized).



$$\mathbb{f} = \begin{pmatrix} 0 \\ 1 \\ 4 \\ 9 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix}$$

$\mathbb{f} = \sum_{i=1}^{4} c_i \mathbb{e}_i$;

$\langle \mathbb{e}_i, \mathbb{e}_i \rangle \, c_i = \sum_{j=1}^{4} f_j \cdot e_i^j$,

where $\mathbb{e}_i^T = (e_i^1, e_i^2, e_i^3, e_i^4)$;

$\mathbb{c} = (c_1, c_2, c_3, c_4) = (3/2, 4, 2, 7/2)$,

$\mathbb{f} = 3/2\, \mathbb{e}_1 + 4\, \mathbb{e}_2 + 2\, \mathbb{e}_3 + 7/2\, \mathbb{e}_4$.

Example: Expanding $\mathbb{f}$ in basis $\{\mathbb{e}_i\}$.

⇒ See next page...

6) SINCE A FUNCTION / VECTOR CAN BE EXPRESSED, EXPANDED AS $\mathbb{f} = \sum_{i=1}^{N} c_i \mathbb{e}_i^n$, THE SET OF COEFFICIENTS $\{c_i\}_{i=1}^{N}$ IS THE SPECTRUM OF $\mathbb{f}$ RELATIVE TO THE MULTI-FREQUENCY ORTHONORMAL BASIS $\{\mathbb{e}_i^n\}_{i=1}^{N}$.

7) IMPLICATION OF FACT 6): GIVEN A (LARGE) SEGMENT OF VOXELS, ONE CAN USE A (RELATIVELY SMALL) SET OF EIGENFUNCTIONS $\{\mathbb{e}_i^n\}$ TO COMPUTE MULTIPLE SPECTRA $\{c_i\}$ FOR VARIOUS LOCAL COMPACT SUBSETS OF THE SEGMENT. BY "COMBINING" THE MULTIPLE LOCAL SPECTRA INTO A SINGLE FEATURE OF THE ENTIRE SEGMENT — e.g., by adding i^{th} coefficient values c_i, for all i and all local spectra — A MULTI-FREQUENCY CHARACTERISTIC OF THE SEGMENT RESULTS.

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions: ... Facts ...

... Continuation of example from previous page:

$$\|\mathbf{e}_i\| = \sqrt{\langle \mathbf{e}_i, \mathbf{e}_i \rangle}$$

$$\Rightarrow \quad \mathbf{e}_i^n = \mathbf{e}_i / \|\mathbf{e}_i\|$$

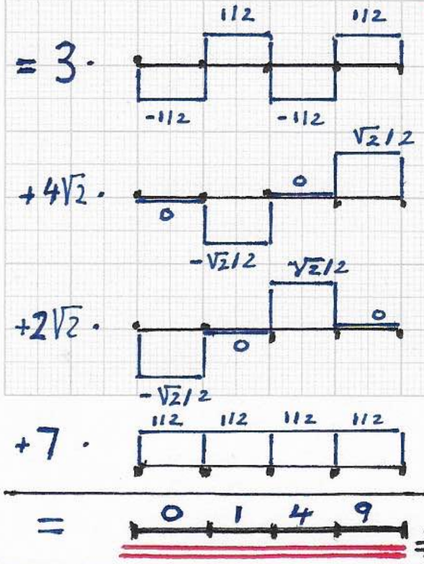
$$\Rightarrow \quad \mathbf{e}_1^n = \mathbf{e}_1/2, \quad \mathbf{e}_2^n = \mathbf{e}_2/\sqrt{2}, \quad \mathbf{e}_3^n = \mathbf{e}_3/\sqrt{2}, \quad \mathbf{e}_4^n = \mathbf{e}_4/2 .$$

$$\mathbf{f} = \sum_{i=1}^{4} c_i \, \mathbf{e}_i^n ; \qquad c_i = \sum_{j=1}^{4} f_j \cdot {}^n e_i^j ,$$

where $\mathbf{e}_i^{n\,T} = ({}^n e_i^1, {}^n e_i^2, {}^n e_i^3, {}^n e_i^4)$;

$$\mathbf{c} = (3, 4\sqrt{2}, 2\sqrt{2}, 7),$$

$$\mathbf{f} = 3\,\mathbf{e}_1^n + 4\sqrt{2}\,\mathbf{e}_2^n + 2\sqrt{2}\,\mathbf{e}_3^n + 7\,\mathbf{e}_4^n ,$$



⇒ Next page...

8) Unlike standard tensor-product-based multi-frequency/-resolution function representations and approximations (e.g., tensor product Fourier or wavelet expansions), the eigenvector/-function approach discussed here makes it possible to expand an "arbitrary-number-of-voxels-based piecewise constant function in a multi-frequency orthonormal basis. The approach can be applied to any number of voxels constituting an arbitrarily shaped segment.

9) The multi-frequency eigenfunction-based expansion can be used to reduce/eliminate noise in a segment by not considering selected high-frequency bands.

10) The eigenfunction-based expansion can be used to perform "inpainting," filling in missing data/holes, by "extrapolating" the expansion into voxels where data are needed.

11) Artifacts in a segment – e.g., streaks – can be removed from a segment: Knowing an artifact's expansion in the eigenfunction basis, one can "subtract" its contribution in the entire segment's expansion and then inpaint the subtraction-induced hole(s).