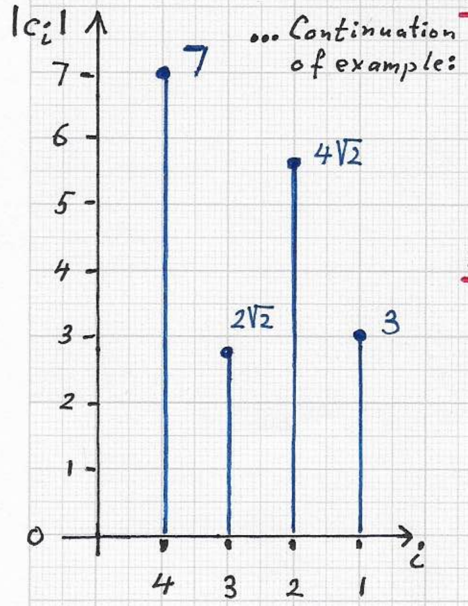
OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions: ... Facts ...

... Continuation of example:



Spectrum of $|c_i|$-values (from low to high frequency).

Normalized eigenvectors for 8-lixel setting (unit mass values, unit spacing), see p. 7, 10/2/2021:

$e_1^n = (-1, 1, -1, 1, -1, 1, -1, 1)^T / 2\sqrt{2}$

$e_2^n = (-\sqrt{2}, 1, 0, -1, \sqrt{2}, -1, 0, 1)^T / 2\sqrt{2}$

$e_3^n = (-1, \sqrt{2}, -1, 0, 1, -\sqrt{2}, 1, 0)^T / 2\sqrt{2}$

$e_4^n = (0, -1, 0, 1, 0, -1, 0, 1)^T / 2$

$e_5^n = (-1, 0, 1, 0, -1, 0, 1, 0)^T / 2$

$e_6^n = (\sqrt{2}, 1, 0, -1, -\sqrt{2}, -1, 0, 1)^T / 2\sqrt{2}$

$e_7^n = (-1, -\sqrt{2}, -1, 0, 1, \sqrt{2}, 1, 0)^T / 2\sqrt{2}$

$e_8^n = (1, 1, 1, 1, 1, 1, 1, 1)^T / 2\sqrt{2}$

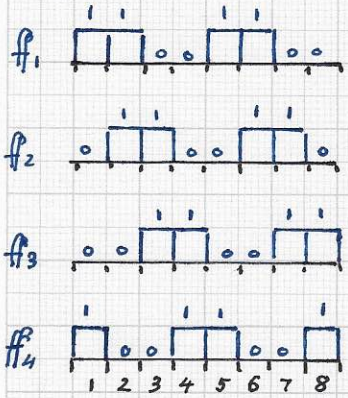
12) With proper data structure design it might be possible to use some coefficients and eigenfunctions of a segment for (lossy) data compression, if desired.

13) When ignoring "minor numerical and discretization artifacts" one can assume that the discussed eigenvalue/-vector approach is invariant to translations and rotations; scaling affects the eigenvalues – but ratios of eigenvalues are invariant to scaling.

Examples. A 4-lixel example is provided on pp. 14-16, 10/7-8/2021. The example shows the computations performed to generate the result, the spectrum of $|c_i|$-values, p. 16, that summarizes how the given 4-lixel function is expressed without loss in the associated periodic eigenbasis $\{e_i^n\}_{i=1}^4$ of normalized eigenfunctions. We also present an 8-lixel example, using normalized eigenfunctions e_i^n, $i=1...8$, for the periodic case. The normalized functions e_i^n, $i=1...8$, are listed here (left).

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions: The chosen 8-lixel example demonstrates that the resulting $|c_i|$-value spectra are invariant to translation. The four functions $f_1, \ldots, f_4$, shown left, represent a "simple periodic texture of pairs of 1-values" – which is translated in the 8-lixel domain.



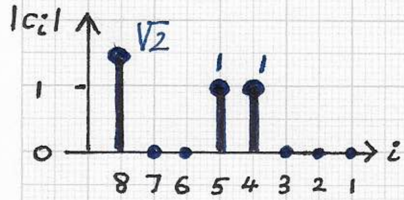
Four 8-lixel functions to be expanded in normalized eigenbasis $\{{}^n e_i\}$, shown on previous page (bottom). The functions f_i are translated versions of each other.

Each 8-lixel function f is expanded as

$$f = \sum_{i=1}^{8} c_i \, {}^n e_i \,, \quad \text{where } c_i = \sum_{j=1}^{8} f_j \cdot {}^n e_i^j \,.$$

Here, the function f is represented by the vector $(f_1, f_2, \ldots, f_8)^T$ and the normalized eigenfunction ${}^n e_i$ is identified with the vector $({}^n e_i^1, {}^n e_i^2, \ldots, {}^n e_i^8)^T$. We summarize the computation of the inner products of f_1 and ${}^n e_i$:

$$c_1 = (-1+1-1+1)/2\sqrt{2} = 0, \quad c_5 = (-1+0-1+0)/2 = -1,$$
$$c_2 = (-\sqrt{2}+1+\sqrt{2}-1)/2\sqrt{2} = 0, \quad c_6 = (\sqrt{2}+1-\sqrt{2}-1)/2\sqrt{2} = 0,$$
$$c_3 = (-1+\sqrt{2}+1-\sqrt{2})/2\sqrt{2} = 0, \quad c_7 = (-1-\sqrt{2}+1+\sqrt{2})/2\sqrt{2} = 0,$$
$$c_4 = (0-1+0-1)/2 = -1, \quad c_8 = (1+1+1+1)/2\sqrt{2} = \sqrt{2}.$$



Spectrum of $|c_i|$-values for functions $f_1, \ldots, f_4$. The four spectra are identical.

Thus, f_1's expansion in the eigenbasis is

$$f_1 = -1\,{}^n e_4 - 1\,{}^n e_5 + \sqrt{2}\,{}^n e_8 \,.$$

Performing the same computations for f_2, f_3 and f_4 yield the expansions for these three functions:

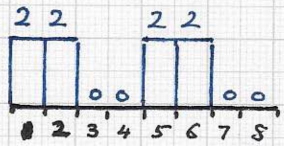
$$f_2 = -1\,{}^n e_4 + 1\,{}^n e_5 + \sqrt{2}\,{}^n e_8 \,,$$
$$f_3 = +1\,{}^n e_4 + 1\,{}^n e_5 + \sqrt{2}\,{}^n e_8 \,,$$
$$f_4 = +1\,{}^n e_4 - 1\,{}^n e_5 + \sqrt{2}\,{}^n e_8 \,.$$

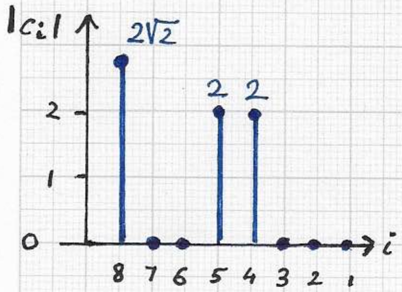
⇒ **INSIGHT:** TRANSLATING A "PERIODIC TEXTURE" IN THE DOMAIN DOES NOT CHANGE ITS $|c_i|$ SPECTRUM.

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions: Two types of scaling must be considered when the effect of scaling on a spectrum is relevant: (i) scaling the function values of a function f by (a global) factor α, i.e., $f \mapsto \alpha f$, and (ii) scaling "the sizes of texture elements (texels)" in the domain of a function f. Case (i) is straightforward: Considering the example involving four 8-lixel functions f_1, f_2, f_3 and f_4, when computing the c_i-values for an expansion of f_1, for example, the c_i coefficients of αf_1 are the c_i coefficients of f_1 multiplied by α. Thus, the $|c_i|$-value spectrum of αf_1 is f_1's spectrum multiplied by α; it is given as coefficient tuple $(\alpha\sqrt{2}, 0, 0, \alpha, \alpha, 0, 0, 0)$, where $\alpha > 0$ and the tuple's components have indices $8, 7, \ldots, 1$ (see figure, left). Case (ii) is explored by considering function f_1 and scaling its "texels" in the 8-lixel domain. The original texel size/width of a texel in f_1's domain is 2; we scale this value by scaling factors $\beta = \frac{1}{2}$ and $\beta = \frac{3}{2}$ and compute the resulting spectra. …

$2f_1$



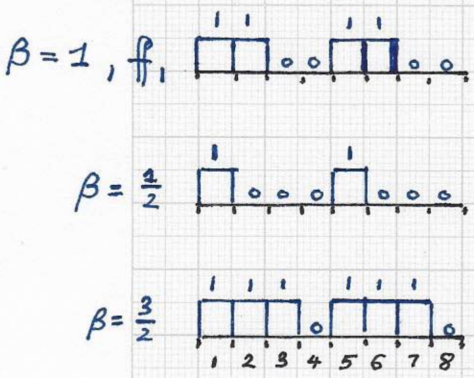


The spectrum of $2f_1$ is the spectrum of f_1, where $|c_i|$-values are multiplied/scaled by 2.

⇒ **INSIGHT:** SCALING THE VALUES OF A FUNCTION f BY A GLOBAL FACTOR α PRODUCES A SPECTRUM THAT IS f'S SPECTRUM WITH ALL COEFFICIENT VALUES MULTIPLIED BY α.

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions: ... The figure (left) shows the three functions to be compared via their spectra: the original function f_1 and two functions where f_1's texels (i.e., their widths in the domain) are scaled with factors $\beta = 1/2$ and $\beta = 3/2$, respectively. The three expansions of these functions in the normalized eigenfunctions are:



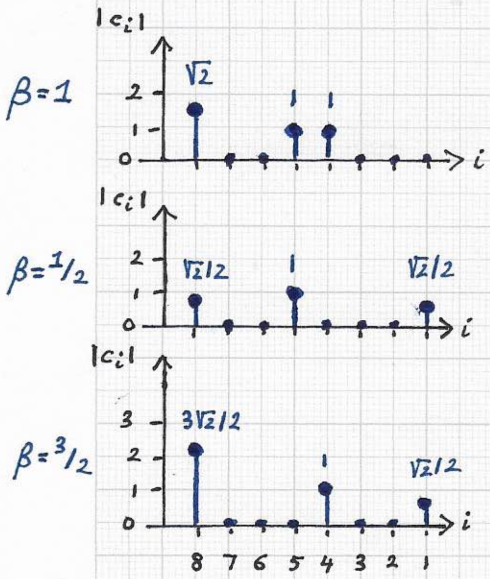
Function f_1 ($\beta = 1$) and two additional functions where the size/width of f_1's "texels" has been scaled by factors $\beta = 1/2$ and $\beta = 3/2$. Note: The number of texels is the same, and also the function values of the texels remain the same (1).

$$\beta = 1: \quad -1\, e_4^n - 1\, e_5^n + \sqrt{2}\, e_8^n ,$$

$$\beta = 1/2: \quad -\sqrt{2}/2\, e_1^n - 1\, e_5^n + \sqrt{2}/2\, e_8^n ,$$

$$\beta = 3/2: \quad -\sqrt{2}/2\, e_1^n - 1\, e_4^n + 3\sqrt{2}/2\, e_8^n .$$

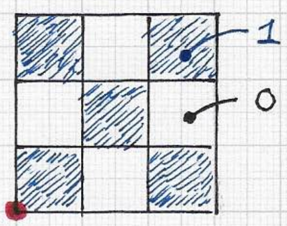
The spectra for these three expansions are shown in the plots (left, bottom). It is not immediately obvious how the scaling of texel width affects the spectra. It is evident that the β-ratios of $1 : \frac{1}{2} : \frac{3}{2}$ are replicated by the c_8 coefficient-ratios $1\sqrt{2} : \frac{1}{2}\sqrt{2} : \frac{3}{2}\sqrt{2}$. This is expected since eigenfunction e_8^n captures a functions overall average value. The effect of texel width scaling on the coefficients $c_1, \ldots, c_7$ is not obvious in this example. Generally, one would expect that a scaling factor $\beta < 1$ ($\beta > 1$) increases the number and values of high-frequency (low-frequency) coefficients c_i.



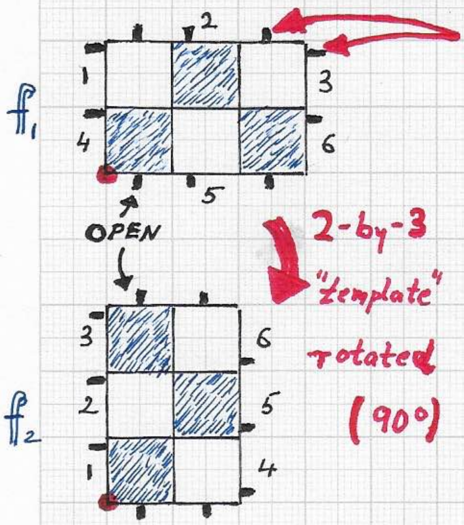
The resulting spectra. Different texel widths generate distinct spectra.

OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions: When processing 2D (pixel) and 3D (voxel) segment data one also concerned about invariance of spectra to rotations/change in orientation. We consider a simple 2-by-3 pixel example, using the eigenvectors/eigenfunctions of the OPEN pixel confuguration shown on p. 12, 10 October 2021, 2nd configuration from the top. The six eigenfunctions provided there define an orthogonal basis, and the normalized versions are:



Given 9-pixel data set.



$$\mathbf{e}_1^n = (-1, \sqrt{2}, -1, 1, -\sqrt{2}, 1)^T / 2\sqrt{2},$$
$$\mathbf{e}_2^n = (1, 0, -1, -1, 0, 1)^T / 2,$$
$$\mathbf{e}_3^n = (1, -\sqrt{2}, 1, 1, -\sqrt{2}, 1)^T / 2\sqrt{2},$$
$$\mathbf{e}_4^n = (-1, -\sqrt{2}, -1, 1, \sqrt{2}, 1)^T / 2\sqrt{2},$$
$$\mathbf{e}_5^n = (-1, 0, 1, -1, 0, 1)^T / 2,$$
$$\mathbf{e}_6^n = (1, \sqrt{2}, 1, 1, \sqrt{2}, 1)^T / 2\sqrt{2}.$$

Normalized eigenfunctions $\mathbf{e}_i^n$ are associated with the i^{th} pixel shown in these two orientations of the underlying 6-pixel "template" used for the representation of the functions $\mathbb{f}_1$ and $\mathbb{f}_2$ in the eigenbasis $\{\mathbf{e}_i^n\}$.

The six-pixel 2-by-3 template has been rotated by 90° around the "origin" ●. (Rotating the template is equivalent to rotating the original data set.)

By expanding $\mathbb{f}_1$ and $\mathbb{f}_2$ in the orthonormal basis $\{\mathbf{e}_i^n\}$ one gains insight into the influence of rotation on spectra.

The general expansion of a "6-pixel function" for this specific pixel configuration and associated orthonormal eigenbasis $\{\mathbf{e}_i^n\}_{i=1}^6$ is $\mathbb{f} = \sum_{i=1}^6 c_i \mathbf{e}_i^n$, where $c_i = \sum_{j=1}^6 f_j \, {}^n e_i^j$. A function $\mathbb{f}$ is represented by the vector $\mathbb{f} = (f_1, f_2, f_3, f_4, f_5, f_6)^T$, and a normalized eigenfunction $\mathbf{e}_i^n$ is identified with the vector $({}^n e_i^1, {}^n e_i^2, \ldots, {}^n e_i^6)^T$. …