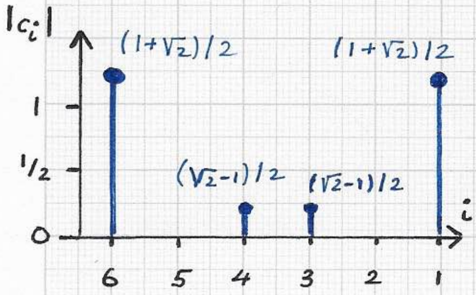
OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions: ... The inner product computations to determine the expansion of ff_1 yield:

$$c_1 = (\sqrt{2}+1+1)/2\sqrt{2} = (2+\sqrt{2})/2\sqrt{2},$$
$$c_2 = (0-1+1)/2 = 0,$$
$$c_3 = (-\sqrt{2}+1+1)/2\sqrt{2} = (2-\sqrt{2})/2\sqrt{2},$$
$$c_4 = (-\sqrt{2}+1+1)/2\sqrt{2} = (2-\sqrt{2})/2\sqrt{2},$$
$$c_5 = (0-1+1)/2 = 0,$$
$$c_6 = (\sqrt{2}+1+1)/2\sqrt{2} = (2+\sqrt{2})/2\sqrt{2}.$$



The resulting spectra for ff_1 and ff_2 are identical (for the considered 90° rotation).

The same computations for ff_2 yield:

$$c_1 = (-2-\sqrt{2})/2\sqrt{2},$$
$$c_2 = 0,$$
$$c_3 = (2-\sqrt{2})/2\sqrt{2},$$
$$c_4 = (-2+\sqrt{2})/2\sqrt{2},$$
$$c_5 = 0,$$
$$c_6 = (2+\sqrt{2})/2\sqrt{2}.$$

⇒ **INSIGHT:** CHANGING THE ORIENTATION OF A SEGMENT, AND THUS ITS TEXTURE IN THE DOMAIN DOES NOT CHANGE ITS $|c_i|$ SPECTRUM.

(BUT: Generally, one must also issues like sampling and resolution; numerical errors; number of eigenfunctions used for an expansion; and number of local spectra computed for one segment.)

Thus, the two resulting expansions are (considering that $(2+\sqrt{2})/2\sqrt{2} = (1+\sqrt{2})/2$):

$$\mathit{ff}_1 = \tfrac{1}{2}(1+\sqrt{2})\,e_1^n + \tfrac{1}{2}(1-\sqrt{2})\,e_3^n + \tfrac{1}{2}(1-\sqrt{2})\,e_4^n + \tfrac{1}{2}(1+\sqrt{2})\,e_6^n,$$

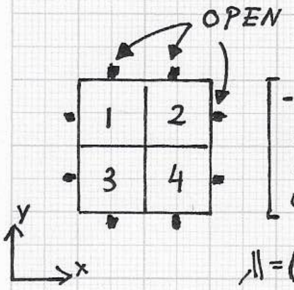
$$\mathit{ff}_2 = \tfrac{1}{2}(-1-\sqrt{2})\,e_1^n + \tfrac{1}{2}(1-\sqrt{2})\,e_3^n + \tfrac{1}{2}(1-\sqrt{2})\,e_4^n + \tfrac{1}{2}(1+\sqrt{2})\,e_6^n.$$

In summary, for this "idealized scenario" of a perfect checker board function with alternating values 0 and 1 and a six-pixel "template" with pixels that are congruent with the pixels of the given function, identical spectra result for rotation angles 0°, 90°, 180° and 270°.

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• **Laplacian eigenfunctions:** When computing local spectra for the eigenfunction expansion of a subset of the pixels (2D case) or voxels (3D case) of a specific-material segment one generally wants to use a pixel domain / voxel domain ("convolution masks") that approximates a perfect disk / ball domain. Thus, the lowest-resolution domains are the 2×2 pixel (2D) or 2×2×2 voxel domains (3D). We briefly summarize the orthonormal eigenbases for the 2D and 3D cases, based on OPEN 2×2 and 2×2×2 eigenfunction templates.

The figures (left) show these simple and "minimalistic" cases of small disk-like and ball-like domains of eigenfunctions. The figures summarize the templates and the used indexing; the $M^{-1}K$ matrices for the OPEN boundary condition; the eigenvalues; and the final normalized mutually orthogonal eigenfunctions.⊛ Selected subsets of a 2D or 3D segment can now be expanded in these eigenbases, producing spectra for the selected 2×2 or 2×2×2 subsets.

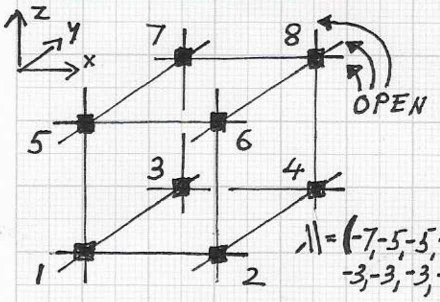


$$\begin{bmatrix} -4 & 1 & 1 & 0 \\ 1 & -4 & 1 & 0 \\ 1 & 0 & -4 & 1 \\ 0 & 1 & 1 & -4 \end{bmatrix}$$

$\Lambda = (-6, -4, -4, -1)$

e_i^n are mutually orthogonal

$$\begin{aligned} e_1^n &= (1, -1, -1, 1)^T / 2, \\ e_2^n &= (-1, 0, 0, 1)^T / \sqrt{2}, \\ e_3^n &= (0, -1, 1, 0)^T / \sqrt{2}, \\ e_4^n &= (1, 1, 1, 1)^T / 2. \end{aligned}$$



$\Lambda = (-7, -5, -5, -5, -3, -3, -3, -1)$

$$\begin{bmatrix} -4 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & -4 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & -4 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & -4 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & -4 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & -4 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & -4 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & -4 \end{bmatrix}$$

$e_i^n, i = 2 \ldots 4,$ $e_i^n, i = 5 \ldots 7,$ are NOT mutually orthogonal!

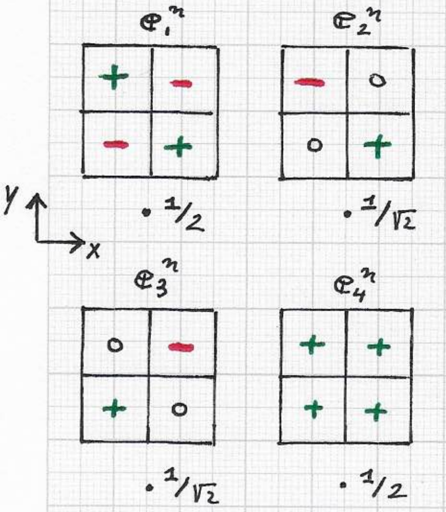
$$\begin{aligned} e_1^n &= (-1, 1, 1, -1, 1, -1, -1, 1)^T / 2\sqrt{2}, \\ e_2^n &= (1, 0, 0, -1, -1, 0, 0, 1)^T / 2, \\ e_3^n &= (0, 1, 0, -1, -1, 0, 1, 0)^T / 2, \\ e_4^n &= (0, 0, 1, -1, -1, 1, 0, 0)^T / 2, \\ e_5^n &= (-1, 0, 0, 1, -1, 0, 0, 1)^T / 2, \\ e_6^n &= (0, -1, 0, -1, 1, 0, 1, 0)^T / 2, \\ e_7^n &= (0, 0, -1, -1, 1, 1, 0, 0)^T / 2, \\ e_8^n &= (1, 1, 1, 1, 1, 1, 1, 1)^T / 2\sqrt{2}. \end{aligned}$$

⊛ Eigenfunctions belonging to different eigenvalues are mutually orthogonal – and only these!

Stratovan

■ OBJECT AND MATERIAL EIGENFUNCTIONS - Cont'd.

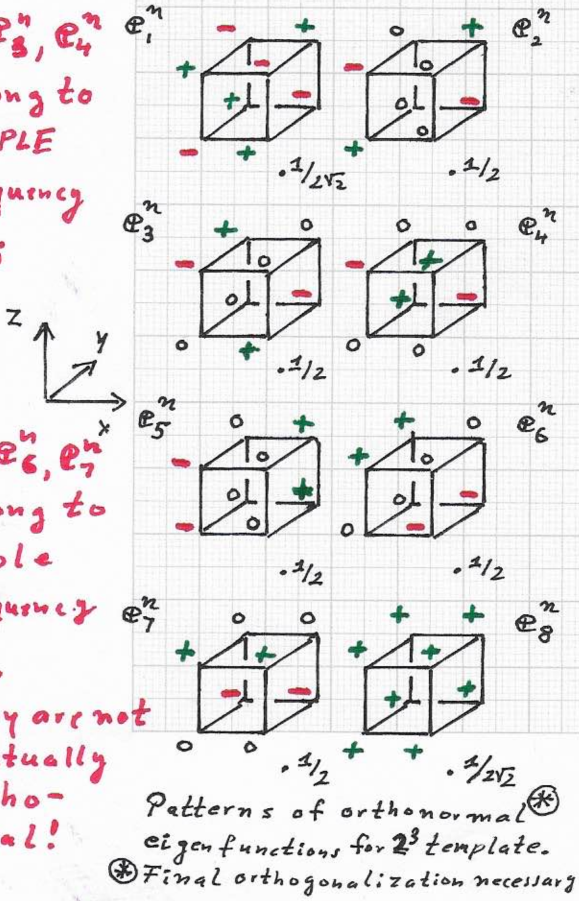
• Laplacian eigenfunctions:



Patterns of orthonormal eigenfunctions for 2×2 pixel template. '+' stands for $+1$, '−' stands for -1, and 'o' represents the value 0.

e_2^n, e_3^n, e_4^n belong to TRIPLE frequency $\sqrt{5}$;

e_5^n, e_6^n, e_7^n belong to triple frequency $\sqrt{3}$. They are not mutually orthogonal!



Patterns of orthonormal eigenfunctions for 2^3 template.(*)

(*) Final orthogonalization necessary!

The 2^2 (2D) and 2^3 (3D) templates are very simple examples of disk-like (2D) and ball-like domains, producing a spectrum for 4 frequencies (2D) and 8 frequencies (3D), respectively. The eigenfunction "patterns" (left) depict how they relate to the frequency behavior of a given function to be expanded in the eigenbasis of orthonormal eigenfunctions. In the 2^2 pixel case, the frequencies ω ($\omega = \sqrt{|\lambda|}$) are $\omega_1 = \sqrt{6}$, $\omega_2 = \omega_3 = 2$, $\omega_4 = 1$; the frequencies in the 2^3 voxel case are $\omega_1 = \sqrt{7}$, $\omega_2 = \omega_3 = \omega_4 = \sqrt{5}$, $\omega_5 = \omega_6 = \omega_7 = \sqrt{3}$, $\omega_8 = 1$. In the 2^2 pixel case, one sees that the frequency $\omega_2 = \omega_3 = 2$ (frequency of multiplicity 2) has two symmetric patterns associated with it: the opposite signs show up on one of the template's diagonals. In the 2^3 voxel case, one observes three symmetric patterns for frequency $\omega_2 = \omega_3 = \omega_4 = \sqrt{5}$ and also for frequency $\omega_5 = \omega_6 = \omega_7 = \sqrt{3}$ (frequencies of multiplicity 3)⊕; here, signs show up at the endpoints of opposite parallel edges (in the figures), with opposite signs at edge endpoints for $\omega_2 = \omega_3 = \omega_4$ and identical signs at edge endpoints for $\omega_5 = \omega_6 = \omega_7$.

⊕ MUST ORTOGONALIZE $\{e_i^n\}_{i=2}^4$ AND $\{e_i^n\}_{i=5}^7$.

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions:

0	1
1	2

(pixel numbering: 1 2 / 3 4)

$\mathbb{f} = (0,1,1,2)^T$

$\mathbb{f} = \sum_{i=1}^{4} c_i \, \mathbb{e}_i^n$

($\mathbb{e}_i^n$: p. 22, 10/13/2021)

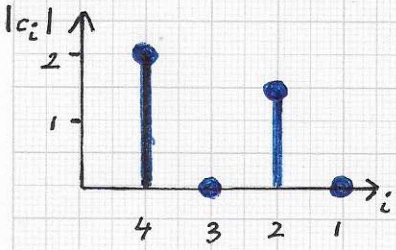
$c_1 = (0-1-1+2)/2 = 0$

$c_2 = (0+2)/\sqrt{2} = \sqrt{2}$

$c_3 = (-1+1)/\sqrt{2} = 0$

$c_4 = (0+1+1+2)/2 = 2$

$\Rightarrow \mathbb{f} = \sqrt{2}\,\mathbb{e}_2^n + 2\,\mathbb{e}_4^n$



Four-pixel function $\mathbb{f}$ expanded via eigenfunctions based on open boundary conditions. The spectrum shows that $\mathbb{f}$ has only one frequency corresponding to detail: ω_2. (Frequency ω_4 captures average – not detail.)

THE EIGENFUNCTIONS $\mathbb{e}_i^n$ ARE NOT MUTUALLY ORTHOGONAL FOR THE 2^3 VOXEL TEMPLATE! ...

It is important to remember that the spectra make it possible to perform a frequency- or "band"-specific analysis and characterization of a given function, i.e., a segment of pixels/voxels with associated function values (e.g., mass values). We consider the simple 2D segment of 2^2 pixels with value $\mathbb{f} = (0,1,1,2)^T$, see left figure. In this simple case, only two LEVELS OF DETAIL, two eigenfunctions, are "present" in $\mathbb{f}$: $\mathbb{f}$ can be expanded (without loss) using only two eigenfunctions, $\mathbb{e}_2^n$ and $\mathbb{e}_4^n$. Here, the AVERAGE BEHAVIOR OF $\mathbb{f}$ is captured in the term $2\,\mathbb{e}_4^n$ and the DETAIL BEHAVIOR OF $\mathbb{f}$ is captured in the term $\sqrt{2}\,\mathbb{e}_2^n$ of the expansion.

We also consider a 3D segment of 2^3 voxels with function value $\mathbb{f} = \left(0, \tfrac{\sqrt{2}}{2}, \tfrac{\sqrt{2}}{2}, -1, \tfrac{\sqrt{2}}{2}, 0, 1, \tfrac{\sqrt{2}}{2}\right)^T$. Since the eigenbasis $\{\mathbb{e}_i^n\}_{i=1}^{8}$ consists of linearly independent, normalized basis functions that are NOT pairwise orthogonal – $\mathbb{e}_2^n, \mathbb{e}_3^n, \mathbb{e}_4^n$ are not orthogonal since $\omega_2 = \omega_3 = \omega_4$ and $\mathbb{e}_5^n, \mathbb{e}_6^n, \mathbb{e}_7^n$ are not orthogonal since $\omega_5 = \omega_6 = \omega_7$ – the computation of the expansion is slightly more expensive.

OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

- Laplacian eigenfunctions:

- 2^3 voxel eigenfunctions:
  i) e_2^n, e_3^n, e_4^n = block,
  ii) e_5^n, e_6^n, e_7^n = block;
  functions in a block NOT mutually orthogonal!
- Computation of c_i-values in expansion $\sum_{i=1}^{8} c_i e_i^n$ requires one to process a BLOCK-DIAGONAL MATRIX: The set $\{c_i\}$ for an expansion of f in the basis $\{e_i^n\}$ is determined by solving a linear system of equations with a block-diagonal matrix: $f = \sum_{i=1}^{8} c_i e_i^n$ is solved via the NORMAL EQUATIONS

$$\begin{pmatrix} \cdots & & \cdots \\ & \langle e_i^n, e_j^n \rangle & \\ \cdots & & \cdots \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_8 \end{pmatrix} = \begin{pmatrix} \vdots \\ \langle f, e_i^n \rangle \\ \vdots \end{pmatrix},$$

$i, j = 1 \dots 8$, where '$\langle \cdot, \cdot \rangle$' denotes the inner product.

Eigenvalues of a real symmetric matrix are real, and eigenvectors belonging to DIFFERENT eigenvalues are orthogonal.

The simple 2^3 voxel example has two eigenvalues of multiplicity 3 ($\omega_2 = \omega_3 = \omega_4$ and $\omega_5 = \omega_6 = \omega_7$ being the consequence); therefore, we have only four "distinct" sets or blocks of eigenfunctions in the 2^3 voxel case: $\{e_1^n\}$, $\{e_2^n, e_3^n, e_4^n\}$, $\{e_5^n, e_6^n, e_7^n\}$, $\{e_8^n\}$. (Note: These four sets have cardinalities 1, 3, 3, 1 – which is the row in Pascal's triangle for the values $\binom{3}{i}$, $i = 0 \dots 3$. This "combinatorial behavior" is a consequence of the fact that $M^{-1}K$ matrix is using a 3D domain space, xyz-space. Thus, the PATTERNS shown on p. 23, 10/14/2021, exhibit the two "symmetry groups" for e_2^n, e_3^n, e_4^n and for e_5^n, e_6^n, e_7^n, with each group having 3 symmetric patterns.) Therefore, a spectrum of the **8 frequencies** $\omega_1, \dots, \omega_8$ "de facto" captures only **4 levels of detail**; the detail-specific coefficient sets of an expansion $f = \sum_{i=1}^{8} c_i e_i^n$ are $\{c_1\}$, $\{c_2, c_3, c_4\}$, $\{c_5, c_6, c_7\}$ and $\{c_8\}$.