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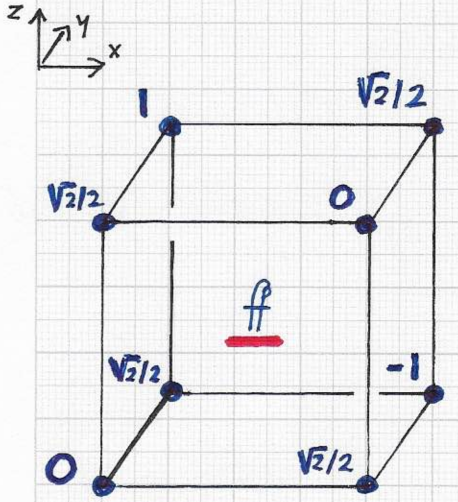
■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• **Laplacian eigenfunctions:** The complete block-diagonal system for the 2^3 voxel template is

$$\begin{bmatrix} \langle 1,1\rangle & \dots & \langle 1,8\rangle \\ \vdots & & \vdots \\ \langle 8,1\rangle & \dots & \langle 8,8\rangle \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_8 \end{bmatrix} = \begin{bmatrix} \langle \mathbf{f},1\rangle \\ \vdots \\ \langle \mathbf{f},8\rangle \end{bmatrix} \Leftrightarrow$$



Synthetically generated 8-voxel function $\mathbf{f}$.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1/2 & 1/2 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 1 & 1/2 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 1/2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1/2 & -1/2 & 0 \\ 0 & 0 & 0 & 0 & -1/2 & 1 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & -1/2 & 1/2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \\ c_7 \\ c_8 \end{bmatrix} = \begin{bmatrix} \langle \mathbf{f},\mathbf{e}_1^n\rangle \\ \langle \mathbf{f},\mathbf{e}_2^n\rangle \\ \langle \mathbf{f},\mathbf{e}_3^n\rangle \\ \langle \mathbf{f},\mathbf{e}_4^n\rangle \\ \langle \mathbf{f},\mathbf{e}_5^n\rangle \\ \langle \mathbf{f},\mathbf{e}_6^n\rangle \\ \langle \mathbf{f},\mathbf{e}_7^n\rangle \\ \langle \mathbf{f},\mathbf{e}_8^n\rangle \end{bmatrix} \circledast ,$$

where $\langle i,j\rangle = \langle \mathbf{e}_i^n, \mathbf{e}_j^n\rangle$ and $\langle \mathbf{f}, i\rangle = \langle \mathbf{f}, \mathbf{e}_i^n\rangle$.

In the context of analysis and expanding the function $\mathbf{f}$, one must determine the unknown coefficients c_i to obtain $\mathbf{f}$'s spectrum in the basis $\{\mathbf{e}_i^n\}$.

⊛ **These inner products are:**

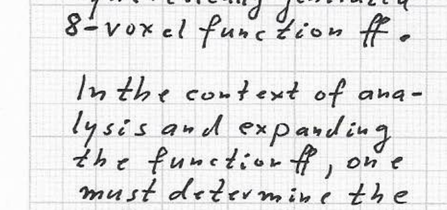
$\langle \mathbf{e}_1^n, \mathbf{f}\rangle = 1$
$\langle \mathbf{e}_2^n, \mathbf{f}\rangle = 1/2$
$\langle \mathbf{e}_3^n, \mathbf{f}\rangle = 1$
$\langle \mathbf{e}_4^n, \mathbf{f}\rangle = 1/2$
$\langle \mathbf{e}_5^n, \mathbf{f}\rangle = -1/2$
$\langle \mathbf{e}_6^n, \mathbf{f}\rangle = 1$
$\langle \mathbf{e}_7^n, \mathbf{f}\rangle = 1/2$
$\langle \mathbf{e}_8^n, \mathbf{f}\rangle = 1$

For simplicity, we present an example where we first create, synthesize, a function $\mathbf{f}$ by specifying a c_i-value tuple:

$$\mathbf{c} = (1,0,1,0,0,1,0,1),$$

$$\mathbf{f} = 1\,\mathbf{e}_1^n + 1\,\mathbf{e}_3^n + 1\,\mathbf{e}_6^n + 1\,\mathbf{e}_8^n$$

$$= \left(0, \sqrt{2}/2, \sqrt{2}/2, -1, \sqrt{2}/2, 0, 1, \sqrt{2}/2\right)^T.$$



Spectrum: Each of the "four levels of detail" is present – with the same "strength" 1.

For function analysis, one must compute the unknown coefficient tuple $\mathbf{c}$ from $\mathbf{f}$ via the block-diagonal system.

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions: Should it be desirable to use a FULLY ORTHOGONAL basis, one can perform the GRAM-SCHMIDT PROCESS to perform a combined orthogonalization-~~normal~~normalization (ORTHONORMALIZATION) to construct a basis $\{\hat{\mathbf{e}}_i^n\}_{i=1}^8$ from the basis $\{\mathbf{e}_i^n\}_{i=1}^8$, where the final basis functions $\hat{\mathbf{e}}_i^n$ are mutually orthogonal to each other **and** are normalized. In our case, **there** are two subsets of basis functions that one must orthonormalize: $\{\mathbf{e}_2^n, \mathbf{e}_3^n, \mathbf{e}_4^n\}$ and $\{\mathbf{e}_5^n, \mathbf{e}_6^n, \mathbf{e}_7^n\}$. (One can apply orthonormalization to these two blocks of three functions each independently – since the two 3-dimensional subspaces of these two blocks are orthogonal to each other, and these subspaces are also orthogonal to the other subspaces of relevance: the subspaces spanned by $\{\mathbf{e}_1^n\}$ and $\{\mathbf{e}_8^n\}$.) (The "Orthogonalize[{{...}, ... {...}}]" function in WolframAlpha can be used.) Orthonormalization yields the new basis $\{\hat{\mathbf{e}}_i^n\}_{i=1}^8$, where $\hat{\mathbf{e}}_1^n = \mathbf{e}_1^n$, $\hat{\mathbf{e}}_8^n = \mathbf{e}_8^n$ and $\{\hat{\mathbf{e}}_2^n, \hat{\mathbf{e}}_3^n, \hat{\mathbf{e}}_4^n\}$ and $\{\hat{\mathbf{e}}_5^n, \hat{\mathbf{e}}_6^n, \hat{\mathbf{e}}_7^n\}$ result when applying orthonormalization to $\{\mathbf{e}_2^n, \mathbf{e}_3^n, \mathbf{e}_4^n\}$ and $\{\mathbf{e}_5^n, \mathbf{e}_6^n, \mathbf{e}_7^n\}$, respectively.

ORTHONORMAL

$\hat{\mathbf{e}}_1^n = (-1, 1, 1, -1, 1, -1, -1, 1)^T / 2\sqrt{2}$,

$\hat{\mathbf{e}}_2^n = (1, 0, 0, -1, -1, 0, 0, 1)^T / 2$,

$\hat{\mathbf{e}}_3^n = (-1, 2, 0, -1, -1, 0, 2, -1)^T / 2\sqrt{3}$,

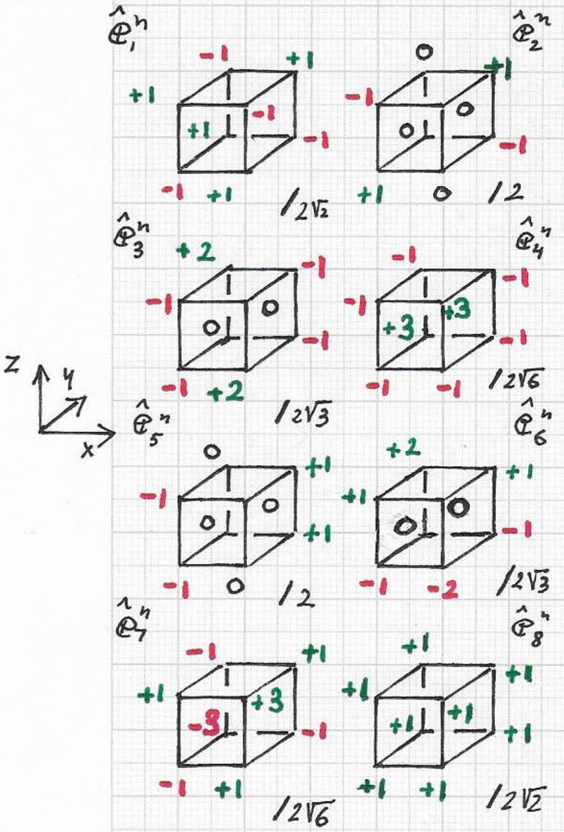
$\hat{\mathbf{e}}_4^n = (-1, -1, 3, -1, -1, 3, -1, -1)^T / 2\sqrt{6}$,

$\hat{\mathbf{e}}_5^n = (-1, 0, 0, 1, -1, 0, 0, 1)^T / 2$,

$\hat{\mathbf{e}}_6^n = (-1, -2, 0, -1, 1, 0, 2, 1)^T / 2\sqrt{3}$,

$\hat{\mathbf{e}}_7^n = (-1, 1, -3, -1, 1, 3, -1, 1)^T / 2\sqrt{6}$,

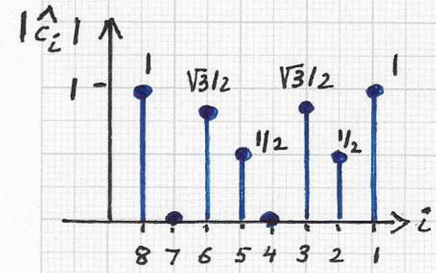
$\hat{\mathbf{e}}_8^n = (1, 1, 1, 1, 1, 1, 1, 1)^T / 2\sqrt{2}$.



Orthonormal basis functions for 2^3 template-functions $\hat{\mathbf{e}}_i^n$, $i=1,\ldots 8$, and their patterns.

■ OBJECT AND MATERIAL EIGENFUNCTIONS — Cont'd.

• **Laplacian eigenfunctions:**



Spectrum: When expanding function $\mathbb{f}$ relative to the orthonormal basis $\{\hat{\mathbb{e}}_i^n\}$ six non-zero $|\hat{c}_i|$-values result.

Depending on the specific classification problem to be solved, one should use the spectrum based on the original eigenbasis $\{\mathbb{e}_i^n\}$ or a guaranteed orthonormal basis $\{\hat{\mathbb{e}}_i^n\}$ — considering a spectrum's capability to represent a material's, a segment's, best possible "fingerprint," "signature."

Nevertheless, since the eigenvalue analysis of the 8-voxel template leads to only four distinct eigenfrequencies (ω_1, $\omega_2=\omega_3=\omega_4$, $\omega_5=\omega_6=\omega_7$, ω_8), both spectral analyses reflect FOUR SCALES of the function $\mathbb{f}$.

While the orthonormalization step enforces mutual orthogonality of all function pairs $\hat{\mathbb{e}}_i^n$, $\hat{\mathbb{e}}_j^n$, $i,j=1\ldots 8$, $i\neq j$, the step destroys the symmetry of the functions in the block $\{\mathbb{e}_2^n, \mathbb{e}_3^n, \mathbb{e}_4^n\}$ and the block $\{\mathbb{e}_5^n, \mathbb{e}_6^n, \mathbb{e}_7^n\}$. Also, the order of orthonormalization steps influences the final set of orthonormal functions in a basis set $\{\hat{\mathbb{e}}_i^n\}_{i=1}^8$.

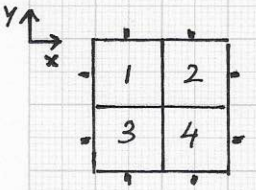
The major advantage of an orthonormal basis is the resulting optimally efficient computation of a function's ($\mathbb{f}$'s) expansion: $\mathbb{f} = \sum_{i=1}^{8} \hat{c}_i \hat{\mathbb{e}}_i^n$ has coefficients $\hat{c}_i$ obtained by computing the inner products $\hat{c}_i = \langle \mathbb{f}, \hat{\mathbb{e}}_i^n \rangle$. Matrix computations are not necessary. We use the function $\mathbb{f} = (0, \sqrt{2}/2, \sqrt{2}/2, -1, \sqrt{2}/2, 0, 1, \sqrt{2}/2)^T$ and compute its associated $\hat{c}_i$ coefficients:

$$\hat{c}_1 = \langle \mathbb{f}, \hat{\mathbb{e}}_1^n \rangle = 1, \quad \hat{c}_2 = \langle \mathbb{f}, \hat{\mathbb{e}}_2^n \rangle = 1/2,$$
$$\hat{c}_3 = \langle \mathbb{f}, \hat{\mathbb{e}}_3^n \rangle = \sqrt{3}/2, \quad \hat{c}_4 = \langle \mathbb{f}, \hat{\mathbb{e}}_4^n \rangle = 0,$$
$$\hat{c}_5 = \langle \mathbb{f}, \hat{\mathbb{e}}_5^n \rangle = -1/2, \quad \hat{c}_6 = \langle \mathbb{f}, \hat{\mathbb{e}}_6^n \rangle = \sqrt{3}/2,$$
$$\hat{c}_7 = \langle \mathbb{f}, \hat{\mathbb{e}}_7^n \rangle = 0, \quad \hat{c}_8 = \langle \mathbb{f}, \hat{\mathbb{e}}_8^n \rangle = 1.$$

$$\Rightarrow \mathbb{f} = 1\,\hat{\mathbb{e}}_1^n + \tfrac{1}{2}\hat{\mathbb{e}}_2^n + \tfrac{\sqrt{3}}{2}\hat{\mathbb{e}}_3^n - \tfrac{1}{2}\hat{\mathbb{e}}_5^n + \tfrac{\sqrt{3}}{2}\hat{\mathbb{e}}_6^n + 1\,\hat{\mathbb{e}}_8^n$$
$$= (0, \sqrt{2}/2, \sqrt{2}/2, -1, \sqrt{2}/2, 0, 1, \sqrt{2}/2)^T.$$

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions:



"open" 2^2-pixel template

$\lambda = (-6, -4, -4, -1)$

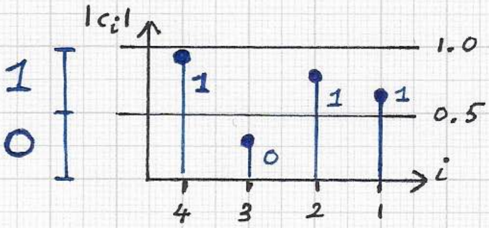
orthonormal eigenfunctions:

$e_1^n = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} / 2 \qquad e_2^n = \begin{bmatrix} -\sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix} / 2$

$e_3^n = \begin{bmatrix} 0 & -\sqrt{2} \\ \sqrt{2} & 0 \end{bmatrix} / 2 \qquad e_4^n = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} / 2$

('/2' coefficients divided by 2)

The four orthonormal eigenfunctions implied by a 2^2-pixel template.



Example of a spectrum and mapping originally real-valued coefficients (normalized to the range [0.0, 1.0]) to either 0 or 1. Here, the resulting "binary quantization" of the $|c_i|$-values produces the "binary spectral tuple" (1, 1, 0, 1) for $(|c_1|, |c_2|, |c_3|, |c_4|)$.

⇒ **Binary quantization of $|c_i|$-values can encode $2^4 = 16$ ($2^8 = 256$) classes in the 2D (3D) case!**

It is important to note that a spectrum, even when only using a small number of eigenfrequencies/eigenfunctions, captures a "large amount of information." This fact is demonstrated by considering the simple 2D example of a small 2^2-pixel template ('open case') and 4 eigenfunctions (left figures). For example, one can use QUANTIZATION when mapping real-valued coefficients c_i to a small number of DISCRETE INTEGER values. When using merely two values (0, 1) to map a real-valued spectrum for four eigenfunctions one can encode $2^4 = 16$ distinct spectra. One can also view such a quantization as a means for "reconstructing an originally given function f in a binary, quantized fashion": $\tilde{f} = \sum_{i=1}^{4} b_i e_i^n$. Here, $b_i \in \{0, 1\}$, i.e., **$\tilde{f}$ is an expansion with integer coefficients b_i, with $b_i = 0$ or $b_i = 1$.** Thus, a function $\tilde{f}$ is only "a very rough approximation" of f; nevertheless, it is possible to capture "16 classes of functions."

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions: Concerning the simple case of a 2^2-pixel template and only two number for quantization, the $2^4 = 16$ possibilities of a function $\tilde{f} = \sum_{i=1}^{4} b_i e_i^n$ represent the ways how one can combine the four eigenfunctions e_i^n, $i = 1 \ldots 4$, using an 'on-bit' (=1) or 'off-bit' (=0). Thus, one also obtains the number 16 when determining the number of these possibilities: $\binom{4}{0}+\binom{4}{1}+\binom{4}{2}+\binom{4}{3}+\binom{4}{4}=16$.

The 16 possibilities to combine 4 functions e_i^n:

	b_1	b_2	b_3	b_4
1	0	0	0	0
4	1	0	0	0
	…			
	0	0	0	1
6	1	1	0	0
	…			
	0	0	1	1
4	1	1	1	0
	…			
	0	1	1	1
1	1	1	1	1

The combinations of two, three and four eigenfunctions generates the following expansions:

$$e_1^n + e_2^n = (1,-1,-1,1)^T/2 + (-\sqrt{2},0,0,\sqrt{2})^T/2 = (1-\sqrt{2},-1,-1,1+\sqrt{2})^T/2,$$

$$e_1^n + e_3^n = (1,-1,-1,1)^T/2 + (0,-\sqrt{2},\sqrt{2},0)^T/2 = (1,-1-\sqrt{2},-1+\sqrt{2},1)^T/2,$$

$$e_1^n + e_4^n = (1,-1,-1,1)^T/2 + (1,1,1,1)^T/2 = (2,0,0,2)^T/2,$$

$$e_2^n + e_3^n = (-\sqrt{2},0,0,\sqrt{2})^T/2 + (0,-\sqrt{2},\sqrt{2},0)^T/2 = (-\sqrt{2},-\sqrt{2},\sqrt{2},\sqrt{2})^T/2,$$

$$e_2^n + e_4^n = (-\sqrt{2},0,0,\sqrt{2})^T/2 + (1,1,1,1)^T/2 = (1-\sqrt{2},1,1,1+\sqrt{2})^T/2,$$

$$e_3^n + e_4^n = (0,-\sqrt{2},\sqrt{2},0)^T/2 + (1,1,1,1)^T/2 = (1,1-\sqrt{2},1+\sqrt{2},1)^T/2,$$

…

$$e_1^n + e_2^n + e_3^n = (1-\sqrt{2},-1-\sqrt{2},-1+\sqrt{2},1+\sqrt{2})^T/2,$$

$$e_1^n + e_2^n + e_4^n = (2-\sqrt{2},0,0,2+\sqrt{2})^T/2,$$

$$e_1^n + e_3^n + e_4^n = (2,-\sqrt{2},\sqrt{2},2)^T/2,$$

$$e_2^n + e_3^n + e_4^n = (1-\sqrt{2},1-\sqrt{2},1+\sqrt{2},1+\sqrt{2})^T/2,$$

$$e_1^n + e_2^n + e_3^n + e_4^n = (2-\sqrt{2},-\sqrt{2},\sqrt{2},2+\sqrt{2})^T/2.$$

…