Bernd Hamann

Stratovan

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions: ... In fact, it is possible to "design" the representation of the histogram vector $\mathbb{h}, \mathbb{h}_1, \mathbb{h}_2, \ldots$ as a problem that leads to an over-determined (or determined) linear system to be solved, i.e., one can "force" $\mathbf{B} \geq \mathbf{k}$. Thus, when the bin resolution $\mathbf{B}$ used for histograms $\mathbb{h}$ and $\mathbb{h}_1, \ldots, \mathbb{h}_k$ is smaller than $\mathbf{k}$, one should increase the value of $\mathbf{B}$ until $\mathbf{B} \geq \mathbf{k}$. For this purpose, an algorithm is needed that has as input a discrete histogram with $\mathbf{B}$ bins and the desired new value of $\mathbf{B}$, generating as output a new, higher-resolution histogram that has the same or very similar characteristics. Important characteristics that must be preserved by such a "re-sampling algorithm" are:

→ preservation of the distribution

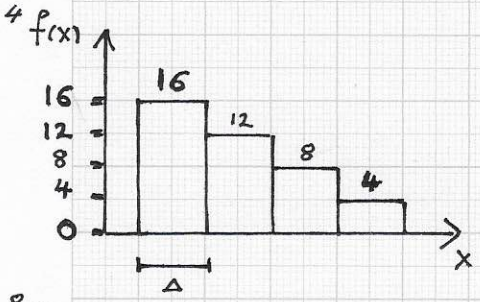
→ preservation of the "normalized nature" of the given histogram

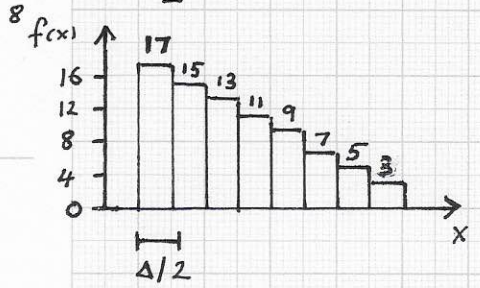
→ enforcement of a "variation-diminishing property," i.e., ensuring that the variation of the new histogram is bounded by that of the given one

→ preservation of certain differential and integral properties

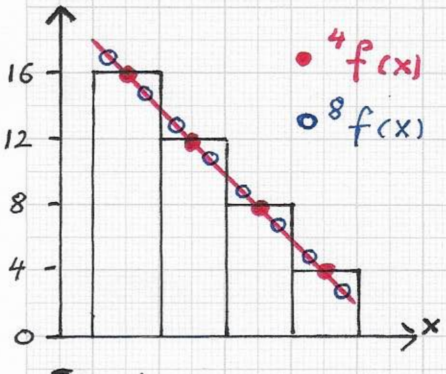
...

• "Re-sampling" piecewise constant function:





Desired behavior of "re-sampling" a piecewise constant function with **4** segments of uniform domain width **Δ** to new piecewise constant function with **8** segments of uniform domain width **Δ/2**.

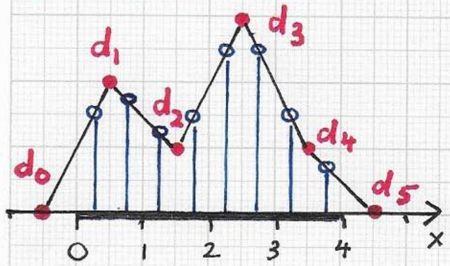


Simple example of increasing resolution from 4 to 8 segments via linear interpolation.

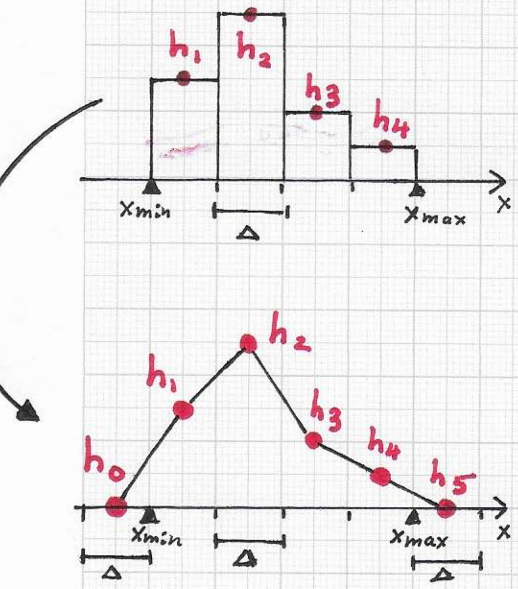
■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Lagrange eigenfunctions: ...

• Linear B-spline:



Original B-spline control polygon consisting of control points $d_0 \ldots d_5$ and sample points ○. In the interval [0,4] the four control points d_1, d_2, d_3 and d_4 are replaced by the eight sample points ○ – effectively doubling the resolution of the discrete approximation of the "underlying shape/function."



A 4-bin histogram with values h_1, h_2, h_3, h_4 defines the control polygon of a B-spline, $h_0, h_1, h_2, h_3, h_4, h_5$.

...Linear and quadratic B-splines are options to consider when increasing the resolution of a given histogram. First, B-splines satisfy the variation diminishing property and thus cannot introduce undesirable oscillations, "undershoots" or "overshoots"; second, they are based on performing convex combinations of the given data (i.e., the B-spline control points) and therefore generate data in the convex hull of the B-spline control points used locally for spline evaluation; third, these low-degree B-splines when used for resampling the given histogram "to a higher resolution" smooth the data by a small degree, which could be helpful for noise reduction; fourth, histogram data (i.e., the numbers of data per bin) must be non-negative, and one can define the control points of a linear/quadratic B-spline in such a way that non-negativity is guaranteed when "sampling to a higher resolution." The principle of (re-) sampling via a linear B-spline is shown in the figure (left). ...

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions: ... We now describe how to double the resolution of a B-bin histogram with values $h_1, h_2, \ldots, h_B$ to a 2B-bin histogram via a linear B-spline re-sampling approach. The input to the algorithm is the set of B histogram values, $h_1, \ldots, h_B$, associated with bins of uniform width Δ. To make sure that the re-sampling process does not produce negative values, the given histogram is artificially "expanded" by a bin 0 to the left of bin 1 and a bin (B+1) to the right of bin B; the values for these two additional bins are $h_0 = h_{B+1} = 0$. The figure (left) shows how the new samples are computed:

$$H_{2i-1} = \frac{1}{4} h_{i-1} + \frac{3}{4} h_i \quad , i = 1 \ldots B$$

$$H_{2i} = \frac{3}{4} h_i + \frac{1}{4} h_{i+1} \quad , i = 1 \ldots B$$

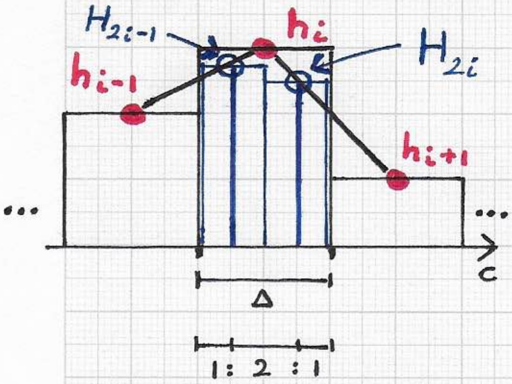
$$\Rightarrow \Delta/2 \left(\sum_{i=1}^{B} H_{2i-1} + H_{2i} \right)$$

$$= \Delta/2 \left(\frac{1}{4} h_0 + \frac{7}{4} h_1 + \frac{8}{4} h_2 + \frac{8}{4} h_3 + \ldots \right.$$
$$\left. \ldots + \frac{8}{4} h_{B-1} + \frac{7}{4} h_B + \frac{1}{4} h_{B+1} \right)$$

$$= \Delta \sum_{i=1}^{B} h_i$$ by setting $h_0 := h_1$ and $h_{B+1} := h_B$.

...

• Re-sampling via a linear B-spline:



Computation of new sample values for H_{2i-1} and H_{2i} via local linear interpolation of h_{i-1} and h_i and of h_i and h_{i+1}.

Original bins have width Δ, bin resulting from re-sampling have width $\Delta/2$.

The INTEGRAL of the piecewise constant function defined by the original histogram is

$$\Delta \sum_{i=1}^{B} h_i .$$

The value of this sum should remain the same after re-sampling.

The integral value is preserved by adding bins 0 and (B+1) of width Δ with associated values $h_0 = h_1$ and $h_{B+1} = h_B$.

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions: ... Re-sampling based on this linear B-spline method produces a new histogram of resolution $2B$ that satisfies the characteristics we expect. **The re-sampling method can be repeated multiple times, generating new histograms of resolutions $2B$, $4B$, $8B$, ...**

We now describe the use of a quadratic B-spline for re-sampling. The illustration (Left) shows the data involved in the computation of the value of H_{2i-1}:

$$a = \frac{3}{8} h_{i-1} + \frac{5}{8} h_i, \qquad b = \frac{7}{8} h_i + \frac{1}{8} h_{i+1}, \qquad H_{2i-1} = \frac{3}{4} a + \frac{1}{4} b$$

$$\Rightarrow \underline{\underline{H_{2i-1}}} = \frac{3}{4}\left(\frac{3}{8} h_{i-1} + \frac{5}{8} h_i\right) + \frac{1}{4}\left(\frac{7}{8} h_i + \frac{1}{8} h_{i+1}\right)$$

$$= \frac{9}{32} h_{i-1} + \frac{22}{32} h_i + \frac{1}{32} h_{i+1}.$$

The value of H_{2i} is obtained via symmetry:

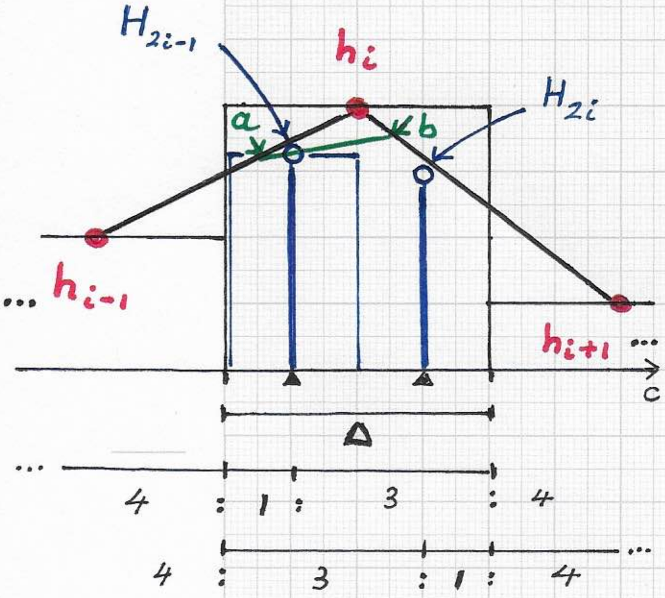
$$\underline{\underline{H_{2i}}} = \frac{1}{4}\left(\frac{1}{8} h_{i-1} + \frac{7}{8} h_i\right) + \frac{3}{4}\left(\frac{5}{8} h_i + \frac{3}{8} h_{i+1}\right)$$

$$= \frac{1}{32} h_{i-1} + \frac{22}{32} h_i + \frac{9}{32} h_{i+1}.$$

Here, $i = 1 \ldots B$, and one must define additional bins 0 and $(B+1)$ with values h_0 and h_{B+1}.

...

• Re-sampling via a quadratic B-spline:



Computation of new sample value for H_{2i-1} (and H_{2i}) via local repeated linear interpolation of h_{i-1}, h_i and h_{i+1}. This computation uses the de Boor algorithm for a quadratic B-spline over a uniform knot partition with interval length Δ.

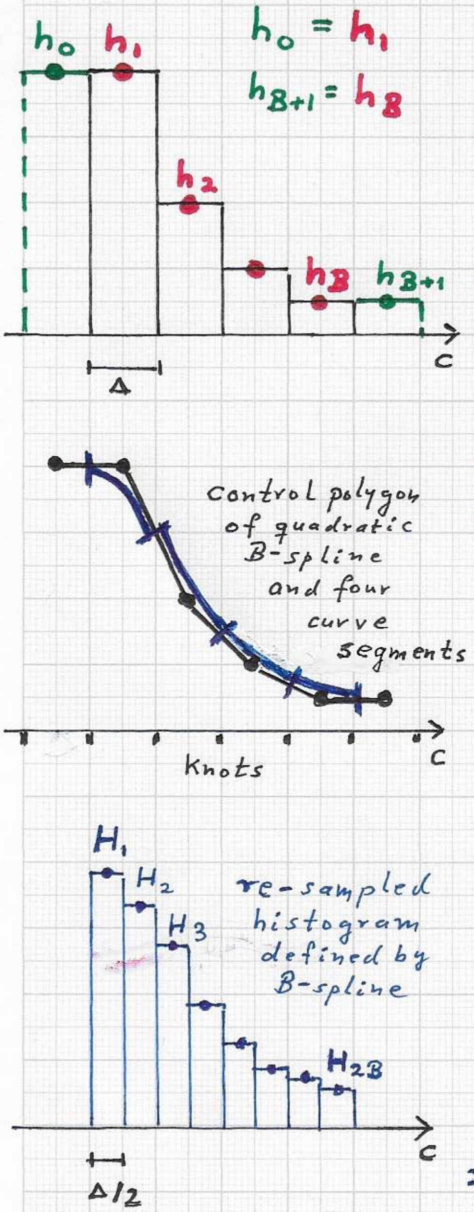
Original bins have uniform bin width Δ, and re-sampling leads to uniform bin width $\Delta/2$. Again, the initial integral value of

$$\Delta \sum_{i=1}^{B} h_i$$

should remain the same after re-sampling.

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions: ... Concerning the definition of the values for h_0 and h_{B+1}, the goal is to preserve the original integral value of the given histogram, i.e., $\Delta \sum_{i=1}^{B} h_i$. Thus, one needs to compute the integral of the new histogram obtained by re-sampling:



Re-sampling via quadratic B-spline method. Top: original histogram; middle: B-spline control polygon and four quadratic curve segments; bottom: re-sampled histogram obtained by evaluating quadratic B-spline.

$$\Delta/2 \left(\sum_{i=1}^{B} H_{2i-1} + H_{2i} \right)$$

$$= \Delta/2 \Big(\tfrac{9}{32} h_0 + \tfrac{22}{32} h_1 + \tfrac{1}{32} h_2$$

$$+ \tfrac{1}{32} h_0 + \tfrac{22}{32} h_1 + \tfrac{9}{32} h_2$$

$$+ \tfrac{9}{32} h_1 + \tfrac{22}{32} h_2 + \tfrac{1}{32} h_3$$

$$+ \tfrac{1}{32} h_1 + \tfrac{22}{32} h_2 + \tfrac{9}{32} h_3$$

$$+ \tfrac{9}{32} h_2 + \tfrac{22}{32} h_3 + \tfrac{1}{32} h_4$$

$$+ \tfrac{1}{32} h_2 + \tfrac{22}{32} h_3 + \tfrac{9}{32} h_4$$

$$+ \tfrac{9}{32} h_3 + \tfrac{22}{32} h_4 + \tfrac{1}{32} h_5$$

$$+ \tfrac{1}{32} h_3 + \tfrac{22}{32} h_4 + \tfrac{9}{32} h_5 + \ldots \Big)$$

$$= \Delta/2 \Big(\tfrac{10}{32} h_0 + \tfrac{54}{32} h_1 + \tfrac{64}{32} h_2 + \ldots +$$

$$+ \tfrac{64}{32} h_{B-1} + \tfrac{54}{32} h_B + \tfrac{10}{32} h_{B+1} \Big)$$

$$= \Delta \Big(\tfrac{5}{32} h_0 + \tfrac{27}{32} h_1 + \sum_{i=2}^{B-1} h_i + \tfrac{27}{32} h_B + \tfrac{5}{32} h_{B+1} \Big)$$

$$= \Delta \sum_{i=1}^{B} h_i$$

by setting $h_0 := h_1$ and $h_{B+1} := h_B$. ...