OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplace eigenfunctions:... The goal is to determine the coefficients α_j of an expansion of a discrete coefficient histogram of a given, new segment using **k stored** discrete coefficient histograms of "samples" of a specific material class – where all histograms initially have the same resolution, the same number $\mathbf{B}$ of bins:

$$\mathbb{h} = \sum_{j=1}^{k} \alpha_j \, \mathbb{h}_j \, .$$

When this linear system is under-determined, i.e., when $\mathbf{B<k}$, one can use the discussed linear or quadratic B-spline based method for re-sampling all discrete histograms. The re-sampling method, as described, makes it possible to repeatedly double the number of bins, from B to $2B$, to $4B$, to $8B$ etc. One terminates the re-sampling step when the number of bins is no longer smaller than k. The values of coefficients α_j can then be computed via the normal equations defined by the over-determined system.

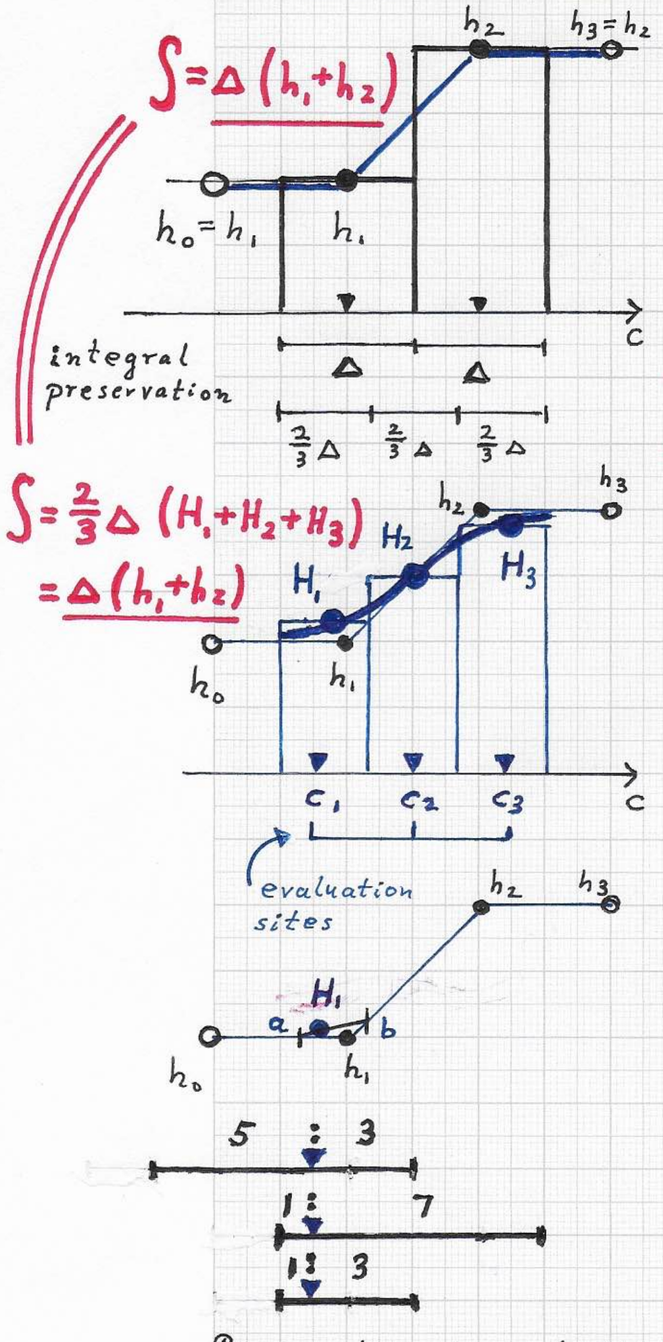
WHEN ONE WANTS TO INCREASE THE NUMBER OF BINS FROM $\mathbf{B}$ TO $\mathbf{2^p B}$, $p \in \{1, 2, \ldots\}$, ONE CAN (i) PERFORM BIN-DOUBLING p TIMES OR (ii) SPLIT EACH ORIGINAL BIN INTO $\mathbf{2^p}$ BINS DIRECTLY....

• Characteristics of re-sampling via linear and quadratic B-splines:

- Re-sampling is a LOCAL method using a small no. of original data.
- Re-sampling computes CONVEX COMBINATIONS of data.
- Thus, re-sampling is VARIATION-DIMINISHING, not introducing undesirable oscillations.
- Re-sampling can be viewed as a smoothing operation.
- The re-sampling method presented preserves the INTEGRALS of histograms.

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions: ... The alternative method to the (repeated) bin-doubling approach is re-sampling performed at bin resolution k. Generally, the solution to the problem of computing the expansion $h = \sum_{j=1}^{k} \alpha_j h_j$ would not require one to handle an under- or over-determined system. A simple example is shown in the figure (left). The given bin resolution of $B = 2$ is increased to resolution 3. Using the quadratic B-spline with control points (ordinates) h_0, h_1, h_2 and h_3 and a "uniform knot partition with spacing Δ," one obtains the 3 histogram values H_1, H_2 and H_3 by evaluating the B-spline at the equidistantly spaced c-values c_1, c_2 and c_3. The computations can be done via the de Boor algorithm (repeated linear interpolation:



Re-sampling via quadratic B-spline. Top: original histogram with 2 bins, defining B-spline control points $h_0, \ldots, h_3$; middle: new histogram with 3 bins and values H_1, H_2 and H_3; bottom: computation of H_1 via de Boor algorithm.

$$H_1 = \frac{3}{4} a + \frac{1}{4} b$$
$$= \frac{3}{4}\left(\frac{3}{8} h_0 + \frac{5}{8} h_1\right) + \frac{1}{4}\left(\frac{7}{8} h_1 + \frac{1}{8} h_2\right)$$
$$= \frac{9}{32} h_0 + \frac{22}{32} h_1 + \frac{1}{32} h_2 = \frac{31}{32} h_1 + \frac{1}{32} h_2 \,,$$

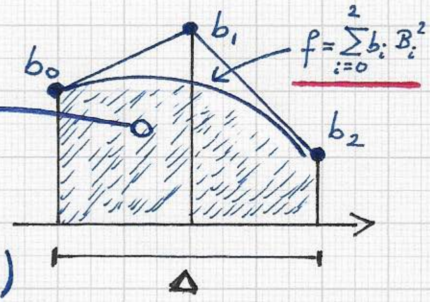
$$H_2 = \frac{1}{2}(h_1 + h_2) \,,$$

$$H_3 = \frac{1}{32} h_1 + \frac{31}{32} h_2 \,. \quad \Rightarrow H_1 + H_2 + H_3 = \frac{3}{2}(h_1 + h_2).$$

...

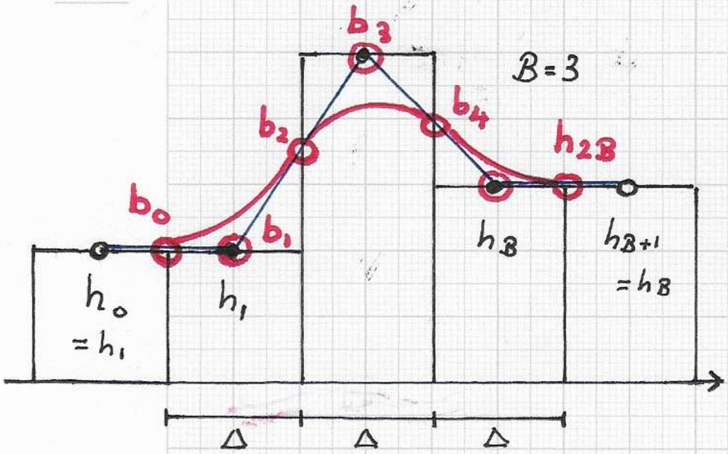
OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions: ... One can show that such a re-sampling approach based on B-splines preserves the integral value of the given histogram for every bin resolution used. In the limit, the new resolution used for re-sampling is ∞, and one can also show that the original histogram's integral value equals the integral value of the continuous B-spline over the same domain interval on the c-axis.



$$\int_\Delta f = \frac{\Delta}{3}(b_0+b_1+b_2)$$

The integral of a function f, a polynomial, expressed via Bernstein polynomials B_i^n over a domain interval Δ has the integral value $\Delta \cdot (b_0+b_1+\ldots+b_n)/(n+1) = \int_\Delta f$.



The figure (left) shows that B bins of width Δ each with histogram values $h_1, \ldots, h_B$ define the $(B+2)$ control points/ordinates of a quadratic B-spline, i.e., $h_0 = h_1$, $h_1, h_2, \ldots, h_B$, $h_{B+1} = h_B$. The quadratic B-spline has B segments; each segment can be represented in the basis of Bernstein polynomials of degree 2 via 3 Bernstein-Bézier control points/coefficients: $b_0, b_1, b_2; b_2, b_3, b_4; \ldots; b_{2B-2}, b_{2B-1}, b_{2B}$. Their values are:

$$b_{2i} = (h_i + h_{i+1})/2, \quad i = 0 \ldots B,$$

$$b_{2i+1} = h_{i+1}, \quad i = 0 \ldots (B-1).$$

Original histogram with B bins of width Δ and values $h_0 = h_1, h_1, \ldots, h_B, h_{B+1} = h_B$; quadratic B-spline control polygon defined by polygon connecting $h_0, h_1, \ldots, h_{B+1}$; each quadratic spline segment converted to Bernstein-Bézier representation defined by a triple of control points/coefficients $b_0, b_1, b_2; b_2, b_3, b_4; \ldots; b_{2B-2}, b_{2B-1}, b_{2B}$.

...

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions: … The integral value of the "piecewise constant histogram function" is $\Delta \sum_{i=1}^{B} h_i$.

The integral value of the quadratic B-spline defined over the same domain with B spline segments is given by

$$\frac{\Delta}{3}\left((b_0 + b_1 + b_2) + \ldots + (b_{2B-2} + b_{2B-1} + b_{2B}) \right).$$

This formula uses the Bernstein-Bézier representation for the B quadratic segments. Using the formulas for $b_0, b_1, \ldots, b_{2B}$ from the previous page, one obtains the integral value as

$$\frac{\Delta}{3}\Big(b_0 + 2b_2 + 2b_4 + \ldots + 2b_{2B-2} + b_{2B} + b_1 + b_3 + b_5 + \ldots + b_{2B-1} \Big)$$

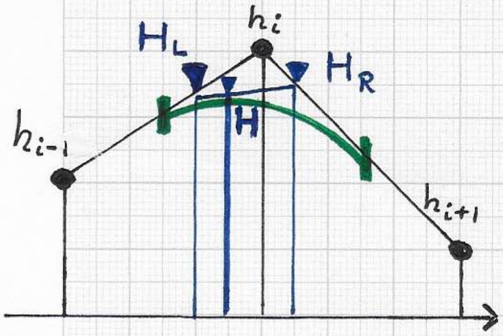
$$= \frac{\Delta}{3}\Big((h_0 + h_1)/2 + (h_1 + h_2) + \ldots + (h_{B-1} + h_B) + (h_B + h_{B+1})/2 + h_1 + h_2 + \ldots + h_B \Big)$$

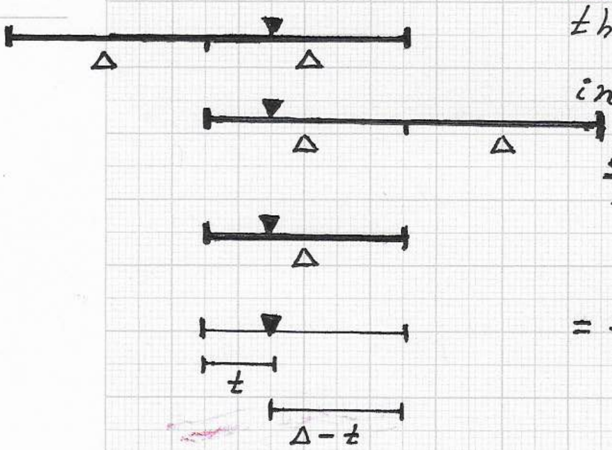
$$= \frac{\Delta}{3}\Big(h_1 + (h_1 + h_2) + \ldots + (h_{B-1} + h_B) + h_B + h_1 + h_2 + \ldots + h_B \Big)$$

$$= \frac{\Delta}{3}\Big(3h_1 + 3h_2 + \ldots + 3h_B \Big) = \Delta \sum_{i=1}^{B} h_i .$$

The implication of this result is that the continuous, analytically defined quadratic B-spline has **the** integral of the "piecewise constant histogram function." …

• **de Boor** algorithm for B-spline evaluation via repeated linear interpolation:





$$H_L = \frac{\Delta - t}{2\Delta} h_{i-1} + \frac{\Delta + t}{2\Delta} h_i$$

$$H_R = \frac{2\Delta - t}{2\Delta} h_i + \frac{t}{2\Delta} h_{i+1}$$

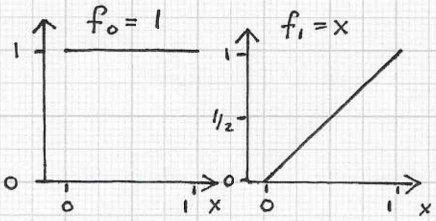
$$H = \frac{\Delta - t}{\Delta} H_L + \frac{t}{\Delta} H_R$$

Using repeated linear interpolation to compute a value H of a quadratic B-spline segment via given histogram values h_{i-1}, h_i and h_{i+1}.

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions: ... In summary, one can employ, for example, the discussed linear or quadratic B-spline based re-sampling approaches to generate new histogram representations with exactly k bins from original histogram data with only B bins ($B < k$). One can use the de Boor algorithm for the calculation of the needed k B-spline values, defined by equidistantly spaced evaluation sites in the histogram's domain and their associated B-spline values. The de Boor algorithm is illustrated on the previous page for the computation of a new value H on a segment of a quadratic B-spline. **Alternatively**, one could even devise an approach using a continuous, analytical B-spline representation for histograms when having to represent a new given histogram as an expansion of stored (material sample) histograms. Using such a completely B-spline, analytical function based approach is discussed next. The shown example (left) provides background. ...

• Analytical approach to function approximation via least-squares:

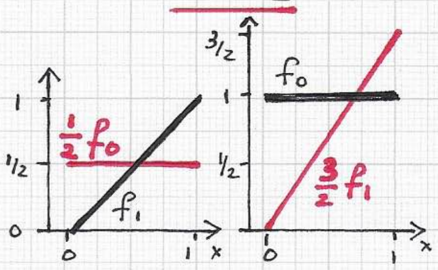


i) best approximation of f_1 in basis $\{f_0\}$:

$$f_1 = c\, f_0$$
$$\langle f_0, f_0 \rangle c = \langle f_0, f_1 \rangle$$
$$1\, c = \tfrac{1}{2}$$
$$c = \tfrac{1}{2}$$

ii) best approximation of f_0 in basis $\{f_1\}$:

$$f_0 = c\, f_1$$
$$\langle f_1, f_1 \rangle c = \langle f_1, f_0 \rangle$$
$$\tfrac{1}{3}\, c = \tfrac{1}{2}$$
$$c = \tfrac{3}{2}$$



Best approximations. Using f_0 to optimally approximate f_1; using f_1 to optimally approximate f_0 (via normal equations).