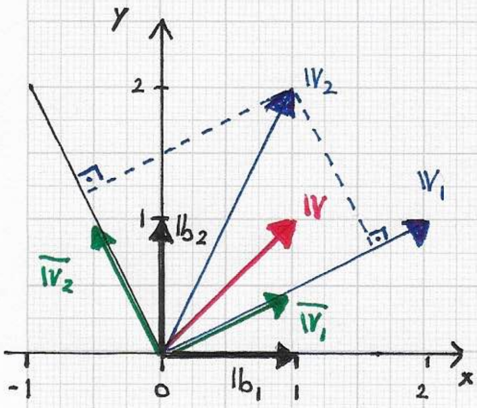
■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions: ... Computational complexity and high efficiency are important issues to consider; practical application of the descrided method is feasible only if a new, given histogram's expansion in terms of k stored sample histograms can be calculated efficiently – even when the value of k (= the value of B, the number of bins) is "large." The set $\{h_i(c)\}$ is assumed to be a basis – consisting of k linearly independent histogram functions $h_i(c)$. Unfortunately, the functions $h_i(c)$ are NOT mutually ORTHOGONAL and are NOT NORMALIZED. For efficient computations, GRAM-SCHMIDT ORTHONORMALIZATION is a method that maps $\{h_i(c)\}$ to an orthonormal basis. Since this method is best explained in a geometrical context, we discuss a simple two-dimensional example first. AN ORTHONORMAL BASIS LEADS TO OPTIMAL EFFICIENCY WHEN REPRESENTING A VECTOR WITH ITS BASIS VECTORS. ...



GRAM-SCHMIDT ORTHONORMALIZATION

Data in example:

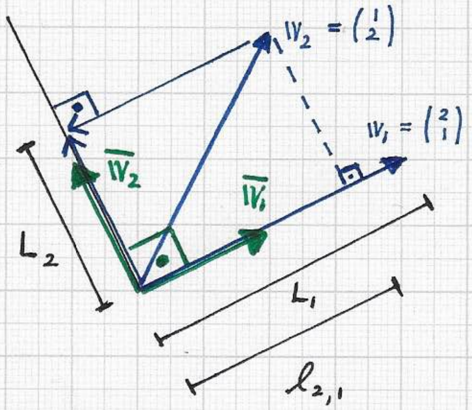
- 2D "canonical" coordinate system with x- and y-axes based on canonical orthonormal basis vectors $\mathbb{b}_1 = (1,0)^T$ and $\mathbb{b}_2 = (0,1)^T$
- Two linearly independent vectors $\mathbb{v}_1 = (2,1)^T$ and $\mathbb{v}_2 = (1,2)^T$
- Vector $\mathbb{v} = (1,1)^T$ to be expressed in the basis $\{\mathbb{v}_1, \mathbb{v}_2\}$
- New ORTHONORMAL basis vectors $\overline{\mathbb{v}_1}$ and $\overline{\mathbb{v}_2}$ resulting from Gram-Schmidt process:

$$\overline{\mathbb{v}_1} = 1/\sqrt{5}\,(2,1)^T$$
$$\overline{\mathbb{v}_2} = 1/\sqrt{5}\,(-1,2)^T$$

Gram-Schmidt orthonormalization computes vectors $\overline{\mathbb{v}_i}$ recursively from given vectors $\mathbb{v}_i$. (See next page.)

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions: ... The figure (left) shows the essential quantities needed to compute the orthonormal basis $\{\overline{\mathbf{w}}_1, \overline{\mathbf{w}}_2\}$:



$$\overline{\mathbf{w}}_1 := \mathbf{w}_1 \,, \qquad \underline{\overline{\mathbf{w}}_1} := \frac{1}{L_1}\, \overline{\mathbf{w}}_1$$

$$\overline{\mathbf{w}}_2 := \mathbf{w}_2 - \langle \mathbf{w}_2, \underline{\overline{\mathbf{w}}_1} \rangle\, \underline{\overline{\mathbf{w}}_1}$$

$$= \mathbf{w}_2 - l_{2,1}\, \underline{\overline{\mathbf{w}}_1} \,,$$

$$\underline{\overline{\mathbf{w}}_2} := \frac{1}{L_2}\, \overline{\mathbf{w}}_2$$

Gram-Schmidt computations:

$$L_1 = \sqrt{5}$$

$$\underline{\overline{\mathbf{w}}_1} := 1/\sqrt{5}\ (2,1)^T$$

$$l_{2,1} = \langle \mathbf{w}_2, \underline{\overline{\mathbf{w}}_1} \rangle = \left\langle \binom{1}{2}, \frac{1}{\sqrt{5}}\binom{2}{1} \right\rangle$$

$$= \frac{2}{\sqrt{5}} + \frac{2}{\sqrt{5}} = 4/\sqrt{5}$$

$$\overline{\mathbf{w}}_2 := \binom{1}{2} - 4/\sqrt{5}\ 1/\sqrt{5} \binom{2}{1}$$

$$= \binom{1}{2} - \binom{8/5}{4/5} = \binom{-3/5}{6/5}$$

$$L_2 = \frac{3}{5}\sqrt{5}$$

$$\underline{\overline{\mathbf{w}}_2} := \frac{1}{L_2}\binom{-3/5}{6/5}$$

$$= 1/\sqrt{5}\binom{-1}{2}$$

WITH RESPECT TO THE NEW ORTHONORMAL BASIS $\{\overline{\mathbf{w}}_1, \overline{\mathbf{w}}_2\}$ THE VECTORS $\mathbf{w}_1$ AND $\mathbf{w}_2$ HAVE COORDINATE TUPLES

$$\boxed{\begin{aligned} \mathbf{w}_{1\,\{\overline{\mathbf{w}}_1, \overline{\mathbf{w}}_2\}} &= \binom{L_1}{0} \\ \mathbf{w}_{2\,\{\overline{\mathbf{w}}_1, \overline{\mathbf{w}}_2\}} &= \binom{l_{2,1}}{L_2} \end{aligned}}$$

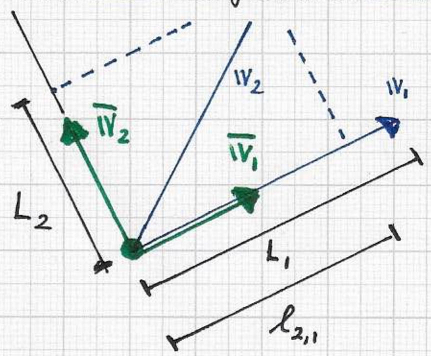
AND .

!

One can view these computations as either iterative or recursive calculation steps, depending on the viewpoint taken. When computing the orthonormal basis one must store the length values $l_{i,j}$ and L_i; they are needed for inverting the Gram-Schmidt process. The Gram-Schmidt process can be described by a matrix (transformation-of-basis matrix), and its inverse matrix is ultimately needed to represent a given vector $\mathbf{w}$ with respect to the non-orthogonal basis $\{\mathbf{w}_1, \mathbf{w}_2\}$. ...

OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions: … In addition to the "underlying" canonical basis $\{\mathbb{b}_1 = (1,0)^T, \mathbb{b}_2 = (0,1)^T\}$, we have the basis $\{\mathbb{v}_1, \mathbb{v}_2\}$ and the constructed orthonormal basis $\{\overline{\mathbb{v}}_1, \overline{\mathbb{v}}_2\}$. WHY HAVE WE CONSTRUCTED THE BASIS $\{\overline{\mathbb{v}}_1, \overline{\mathbb{v}}_2\}$? WITH RESPECT TO $\{\overline{\mathbb{v}}_1, \overline{\mathbb{v}}_2\}$ WE CAN COMPUTE A REPRESENTATION OF A GIVEN VECTOR $\mathbb{v}$ IN **MINIMAL TIME**. IS $\mathbb{v}$'S REPRESENTATION WITH RESPECT TO $\{\overline{\mathbb{v}}_1, \overline{\mathbb{v}}_2\}$ THE EVENTUAL GOAL? NO! THE EVENTUAL GOAL IS THE **REPRESENTATION OF $\mathbb{v}$ WITH RESPECT TO $\{\mathbb{v}_1, \mathbb{v}_2\}$**. Thus:



i) Compute $\{\overline{\mathbb{v}}_1, \overline{\mathbb{v}}_2\}$.

ii) Store the length values $l_{i,j}$ and L_i resulting from the Gram-Schmidt process – producing a triangle matrix T. The inverse of T (T^{-1}) is a triangle matrix as well. Store T^{-1}.

iii) Given a vector $\mathbb{v}$ – originally represented with respect to the canonical basis $\{\mathbb{b}_1, \mathbb{b}_2\}$ – represent it w.r.t. the basis $\{\overline{\mathbb{v}}_1, \overline{\mathbb{v}}_2\}$.

iv) Use T^{-1} to compute the desired **REPRESENTATION OF $\mathbb{v}$ w.r.t. $\{\mathbb{v}_1, \mathbb{v}_2\}$**.

Basis transformation.

The matrix that describes the mapping of basis vectors $\overline{\mathbb{v}}_1$ to $\mathbb{v}_1$ and $\overline{\mathbb{v}}_2$ to $\mathbb{v}_2$ is the triangle matrix

$$T = \begin{bmatrix} L_1 & l_{2,1} \\ 0 & L_2 \end{bmatrix}.$$

The columns of T are the representations of $\mathbb{v}_1$ and $\mathbb{v}_2$ relative to the orthonormal basis vectors $\overline{\mathbb{v}}_1$ and $\overline{\mathbb{v}}_2$.

The inverse of T is also a triangle matrix; it is

$$T^{-1} = \begin{bmatrix} 1/L_1 & -l_{2,1}/(L_1 L_2) \\ 0 & 1/L_2 \end{bmatrix}.$$

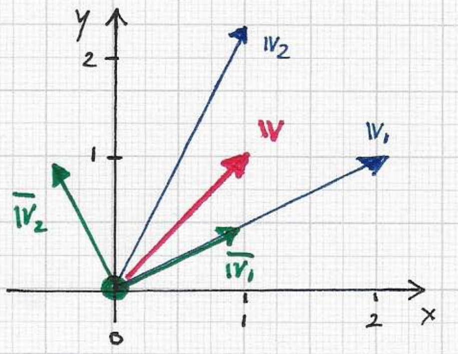
THE REPRESENTATION OF A VECTOR $\mathbb{v}$ W.R.T. $\{\mathbb{v}_1, \mathbb{v}_2\}$ IS OBTAINED FROM ITS REPRESENTATION W.R.T. $\{\overline{\mathbb{v}}_1, \overline{\mathbb{v}}_2\}$ AS FOLLOWS:

$$\boxed{\mathbb{v}_{\{\mathbb{v}_1, \mathbb{v}_2\}} = T^{-1}\, \mathbb{v}_{\{\overline{\mathbb{v}}_1, \overline{\mathbb{v}}_2\}}}.$$

…

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions: ...



The triangular matrices for this specific example are:

$$T = \begin{bmatrix} \sqrt{5} & 4/\sqrt{5} \\ 0 & 3\sqrt{5}/5 \end{bmatrix},$$

$$T^{-1} = \begin{bmatrix} 1/\sqrt{5} & -4/3\sqrt{5} \\ 0 & 5/3\sqrt{5} \end{bmatrix}.$$

$\{\bar{\mathbf{v}}_1, \bar{\mathbf{v}}_2\}$ is orthonormal $\Rightarrow$

$$\mathbf{v}_{\{\bar{\mathbf{v}}_1, \bar{\mathbf{v}}_2\}} = \begin{pmatrix} \langle \mathbf{v}, \bar{\mathbf{v}}_1 \rangle \\ \langle \mathbf{v}, \bar{\mathbf{v}}_2 \rangle \end{pmatrix} = \begin{pmatrix} \left\langle \begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}2\\1\end{pmatrix}\cdot\frac{1}{\sqrt{5}} \right\rangle \\ \left\langle \begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}-1\\2\end{pmatrix}\cdot\frac{1}{\sqrt{5}} \right\rangle \end{pmatrix} = \begin{pmatrix} 3/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix}.$$

Mapping via T^{-1} $\Rightarrow$

$$\mathbf{v}_{\{\mathbf{v}_1, \mathbf{v}_2\}} = T^{-1}\, \mathbf{v}_{\{\bar{\mathbf{v}}_1, \bar{\mathbf{v}}_2\}} = \begin{pmatrix} 1/\sqrt{5} & -4/3\sqrt{5} \\ 0 & 5/3\sqrt{5} \end{pmatrix} \begin{pmatrix} 3/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix} = \begin{pmatrix} 3/5 - 4/15 \\ 1/3 \end{pmatrix} = \begin{pmatrix} 1/3 \\ 1/3 \end{pmatrix}.$$

$\Rightarrow$ **Test:** $\frac{1}{3}\mathbf{v}_1 + \frac{1}{3}\mathbf{v}_2 = \begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix} + \begin{pmatrix} 1/3 \\ 2/3 \end{pmatrix} = (1,1)^T$ ✓

We briefly discuss the three-dimsional and general **k**-dimensional cases as our driving application concerns the expansion of a given **k**-dimensional "histogram function" in terms of a non-orthoganal basis of stored sample **k**-dimensional "histogram funetions" that are linearly independent and not normalized.

In the geometrical setting, these are the necessary **Gram-Schmidt orthonormalization** steps mapping the basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ to the orthonormal basis $\{\bar{\mathbf{v}}_1, \bar{\mathbf{v}}_2, \bar{\mathbf{v}}_3\}$:

$$\bar{\mathbf{v}}_1 := \mathbf{v}_1, \quad \bar{\mathbf{v}}_1 := \frac{1}{\|\bar{\mathbf{v}}_1\|}\bar{\mathbf{v}}_1 = \frac{1}{L_1}\bar{\mathbf{v}}_1$$

$$\bar{\mathbf{v}}_2 := \mathbf{v}_2 - \langle \mathbf{v}_2, \bar{\mathbf{v}}_1 \rangle \bar{\mathbf{v}}_1 = \mathbf{v}_2 - l_{2,1}\bar{\mathbf{v}}_1,$$

$$\bar{\mathbf{v}}_2 := \frac{1}{\|\bar{\mathbf{v}}_2\|}\bar{\mathbf{v}}_2 = \frac{1}{L_2}\bar{\mathbf{v}}_2$$

$$\bar{\mathbf{v}}_3 := \mathbf{v}_3 - \langle \mathbf{v}_3, \bar{\mathbf{v}}_1 \rangle \bar{\mathbf{v}}_1 - \langle \mathbf{v}_3, \bar{\mathbf{v}}_2 \rangle \bar{\mathbf{v}}_2 = \mathbf{v}_3 - l_{3,1}\bar{\mathbf{v}}_1 - l_{3,2}\bar{\mathbf{v}}_2,$$

$$\bar{\mathbf{v}}_3 := \frac{1}{\|\bar{\mathbf{v}}_3\|}\bar{\mathbf{v}}_3 = \frac{1}{L_3}\bar{\mathbf{v}}_3$$

The resulting triangular matrix T is

$$T = \begin{bmatrix} L_1 & l_{2,1} & l_{3,1} \\ 0 & L_2 & l_{3,2} \\ 0 & 0 & L_3 \end{bmatrix}. \ \ldots$$

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions: ... The desired inverse matrix of T is

$$T^{-1} = \begin{bmatrix} \frac{1}{L_1} & -\frac{l_{2,1}}{L_1 L_2} & \frac{l_{2,1} l_{3,2} - L_2 l_{3,1}}{L_1 L_2 L_3} \\ 0 & \frac{1}{L_2} & -\frac{l_{3,2}}{L_2 L_3} \\ 0 & 0 & \frac{1}{L_3} \end{bmatrix} .$$

Gram-Schmidt orthonormalization applied to a k-dimensional basis $\{\mathbb{V}_1, \ldots, \mathbb{V}_k\}$ generating the orthonormal basis $\{\overline{\mathbb{V}}_1, \ldots, \overline{\mathbb{V}}_k\}$ can be summarized as follows:

$$\overline{\mathbb{V}}_1 := \mathbb{V}_1 \quad , \quad \overline{\mathbb{V}}_1 := \frac{1}{\|\overline{\mathbb{V}}_1\|} \overline{\mathbb{V}}_1 = \frac{1}{L_1} \overline{\mathbb{V}}_1$$

$$\overline{\mathbb{V}}_i := \mathbb{V}_i - \sum_{j=1}^{i-1} \langle \mathbb{V}_i, \overline{\mathbb{V}}_j \rangle \overline{\mathbb{V}}_j \quad ,$$

$$\overline{\mathbb{V}}_i := \frac{1}{\|\overline{\mathbb{V}}_i\|} \overline{\mathbb{V}}_i = \frac{1}{L_i} \overline{\mathbb{V}}_i \quad ;$$

$$i = 2, \ldots, k .$$

The resulting triangular matrix is

$$T = \begin{bmatrix} L_1 & l_{2,1} & l_{3,1} & \ldots & l_{k,1} \\ 0 & L_2 & l_{3,2} & \ldots & l_{k,2} \\ 0 & 0 & L_3 & \ldots & \vdots \\ \vdots & \vdots & \vdots & & \\ 0 & 0 & 0 & \ldots & L_k \end{bmatrix} .$$

• **NOTE.** The values of L_i and $l_{j,i}$ are computed via Gram-Schmidt orthonormalization as "by-products." These values are stored as matrix T.

...

• **Computational complexity:**

→ A given k-dimensional vector $\mathbb{V}$ must be represented with respect to a basis $\{\mathbb{V}_1, \ldots, \mathbb{V}_k\}$.

→ An orthonormal basis $\{\overline{\mathbb{V}}_1, \ldots, \overline{\mathbb{V}}_k\}$ is known; it has been pre-computed.

→ The triangular matrix T^{-1} is known; it also has been pre-computed.

1) One computes the k inner products $\langle \mathbb{V}, \overline{\mathbb{V}}_i \rangle$, $i = 1 \ldots k$. THE TOTAL NUMBER OF MULTIPLICATIONS IS $k * k$.

2) One multiplies matrix T^{-1} and the representation of vector $\mathbb{V}$ obtained by step **1)**. Since T^{-1} is an upper triangular matrix, THE TOTAL NUMBER OF MULTIPLICATIONS IS $(k*k+k)/2$.

⇒ THE OVERALL NUMBER OF MULTIPLICATIONS PERFORMED TO OBTAIN $\mathbb{V}$'S REPRESENTATION WITH RESPECT TO $\mathbb{V}_1, \ldots, \mathbb{V}_k$ IS

$$\frac{3}{2} k^2 + \frac{1}{2} k .$$

!