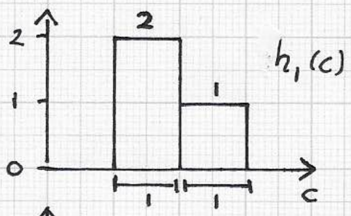
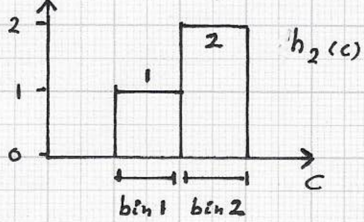
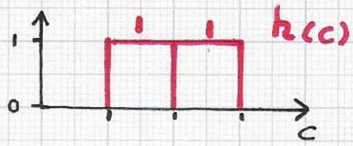
■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions: …

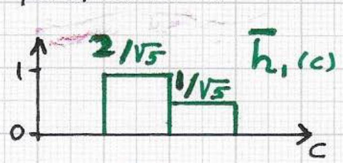


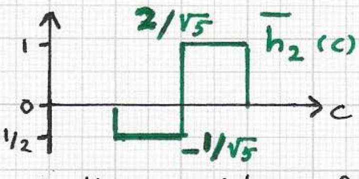




"Transfer" of the geometrical method to the analysis of functions.

The functions considered are "histogram functions," i.e., piecewise constant functions defined over k uniform bins of unit width.





Orthonormal basis functions computed as:

$\bar{h}_1(c) := h_1(c) / \|h_1(c)\|$

$\bar{h}_2(c) := h_2(c) - \langle h_2(c), \bar{h}_1(c) \rangle \bar{h}_1(c),$

$\bar{h}_2(c) := \bar{h}_2(c) / \|\bar{h}_2(c)\|$

The matrix T is an upper triangular matrix; its inverse T^{-1} can be computed via back substitution in an efficient manner. The values of all elements of T^{-1} are pre-computed and stored for eventual usage.

$$\boxed{\text{THE TWO-STEP PROCESS OF REPRESENTING A } k\text{-DIMENSIONAL VECTOR } \mathbf{v} \text{ WITH RESPECT TO A PRE-COMPUTED ORTHONORMAL BASIS AND THEN APPLYING THE PRE-COMPUTED TRIANGULAR MATRIX } T^{-1} \text{ IS OF COMPLEXITY } O(k^2) \text{, CONSIDERING ALL MULTIPLICATIONS INVOLVED.}}$$

The naive approach for computing the representation of a given k-dimensional vector $\mathbf{v}$ with respect to a non-orthogonal basis is of complexity $O(k^3)$. For example, if $k = 1000$ (= number of samples, number of sample histogram functions), then the described method will be 1000 times faster than the naive approach. …

OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions: ... We must "transfer" the described method to our driving application: "histogram functions." The figures shown on the previous page show the two-dimensional example, discussed in detail in the geometrical setting, for the scenario of "histogram functions": Given is function $h(c)$ that must be represented with respect to non-orthogonal functions $h_1(c)$ and $h_2(c)$. All functions have the same domain, consisting of two bins of uniform unit width and constant function value per bin. The resulting Gram-Schmidt orthonormalization computations are summarized on this page (left). The upper triangular matrices are again

$$T = \begin{bmatrix} \sqrt{5} & 4/\sqrt{5} \\ 0 & 3\sqrt{5}/5 \end{bmatrix} \quad \text{and} \quad T^{-1} = \begin{bmatrix} 1/\sqrt{5} & -4/3\sqrt{5} \\ 0 & 5/3\sqrt{5} \end{bmatrix}.$$

One obtains:

$$h(c)_{\{\bar{h}_1(c), \bar{h}_2(c)\}} = \langle h(c), \bar{h}_1(c) \rangle \bar{h}_1(c) + \langle h(c), \bar{h}_2(c) \rangle \bar{h}_2(c) = 3/\sqrt{5}\, \bar{h}_1(c) + 1/\sqrt{5}\, \bar{h}_2(c) ;$$

...

• Example from previous page – "histogram functions":

$$\langle h_2(c), \bar{h}_1(c) \rangle = = \int_{\substack{\text{all}\\ \text{bins}}} h_2(c) \bar{h}_1(c)\, dc ,$$

$$\| h_1(c) \| = \left(\int_{\substack{\text{all}\\ \text{bins}}} (h_1(c))^2 dc \right)^{1/2}$$

$$\| \bar{h}_2(c) \| = \ldots$$

$$\Rightarrow \bar{h}_1(c) := \frac{1}{\sqrt{5}} h_1(c) ,$$

$$\bar{h}_2(c) := h_2(c) - \frac{4}{\sqrt{5}} \bar{h}_1(c) = h_2(c) - \frac{4}{5} h_1(c)$$

$$\Rightarrow \| \bar{h}_2 \| = \frac{3}{5}\sqrt{5}$$

$$\Rightarrow \bar{h}_2(c) := \left(h_2(c) - \frac{4}{5} h_1(c) \right) \cdot \frac{5}{3\sqrt{5}} = \frac{1}{\sqrt{5}} \left(\frac{5}{3} h_2(c) - \frac{4}{3} h_1(c) \right) = \frac{1}{3\sqrt{5}} \left(5 h_2(c) - 4 h_1(c) \right)$$

$$\Rightarrow \bar{h}_1(c) = \begin{cases} 2/\sqrt{5} , & \text{bin } 1 \\ 1/\sqrt{5} , & \text{bin } 2 \end{cases}$$

$$\bar{h}_2(c) = \begin{cases} -1/\sqrt{5} , & \text{bin } 1 \\ 2/\sqrt{5} , & \text{bin } 2 \end{cases}$$

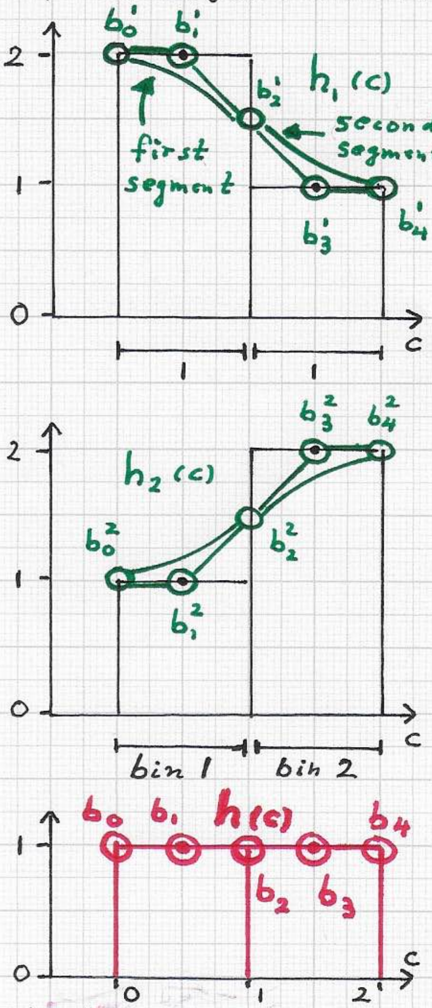
■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions: … $h(c)_{\{h_1(c),\, h_2(c)\}} =$

$$= (h_1(c)\;\; h_2(c)) \begin{bmatrix} 1/\sqrt{5} & -4/3\sqrt{5} \\ 0 & 5/3\sqrt{5} \end{bmatrix} \begin{bmatrix} 3/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$$

$$= (h_1(c)\;\; h_2(c)) \begin{pmatrix} 1/3 \\ 1/3 \end{pmatrix} = \frac{1}{3} h_1(c) + \frac{1}{3} h_2(c).$$

$$\Rightarrow h(c) = \sum_{i=1}^{2} \alpha_i h_i(c), \quad \alpha = \left(\frac{1}{3}, \frac{1}{3}\right).$$



Using quadratic B-splines to represent stored "histogram functions" $h_1(c)$ and $h_2(c)$ and new, given function $h(c)$. All functions have the same domain consisting of the same number of uniform unit bins on the c-axis. Each B-spline segment is represented in Bernstein–Bézier form with three control points/control coefficients shown as ⭘ and ⭘.

This example shows that the Gram–Schmidt orthonormalization approach can be employed for computing a given "histogram function's" representation with respect to a stored non-orthogonal basis of non-orthogonal functions highly efficiently.

For completeness and as a simple more general example, we present the discussed "histogram function" scenario for quadratic B-splines. In this setting, a quadratic B-spline segment is constructed over all bins that define the same domain of all "histogram functions." The figure (left) illustrates the setting. The function $h(c)$ must be expanded in the basis $\{h_1(c), h_2(c)\}$. …

OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions: ... **The** function $h(c)$ used in this example is a constant quadratic B-spline "histogram function" of value 1; thus, all its Bernstein–Bézier coefficients $b_0, \ldots, b_4$ have the value 1. The used "histogram basis functions" are not orthogonal to each other; they are called $h_1(c)$ and $h_2(c)$ with coefficients $b_0^1, \ldots, b_4^1$ and $b_0^2, \ldots, b_4^2$, respectively. In this simple scenario, one "sees" that $h(c) = \frac{1}{3} h_1(c) + \frac{1}{3} h_2(c)$, since the coefficients satisfy the linear relationship $b_i = \frac{1}{3} b_i^1 + \frac{1}{3} b_i^2$, $i = 0 \ldots 4$. We now perform Gram–Schmidt orthonormalization and the subsequent basis transformation to derive the representation of $h(c)$ formally.

1. Normalization of $h_1(c)$

$$\|h_1(c)\| = \left(\int_0^2 h_1(c)^2 \right)^{1/2} = \left(\int_0^1 h_1(c)^2 + \int_1^2 h_1(c)^2 \right)^{1/2}$$

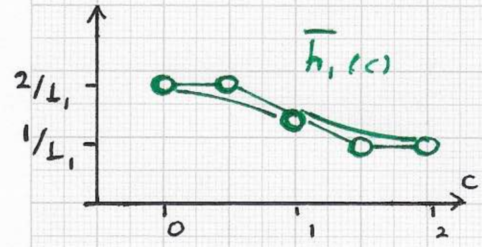
$$[0,1]: \ h_1(c)^2 = 4 B_0^4(c) + 4 B_1^4(c) + \frac{11}{3} B_2^4(c) + 3 B_3^4(c) + \frac{9}{4} B_4^4(c)$$

$$\Rightarrow \int_0^1 h_1(c)^2 = \frac{1}{5}\left(4 + 4 + \frac{11}{3} + 3 + \frac{9}{4}\right) = \frac{203}{60}$$

$$[1,2]: \ h_1(c)^2 = \frac{9}{4} B_0^4(c) + \frac{3}{2} B_1^4(c) + \frac{7}{6} B_2^4(c) + 1 B_3^4(c) + 1 B_4^4(c)$$

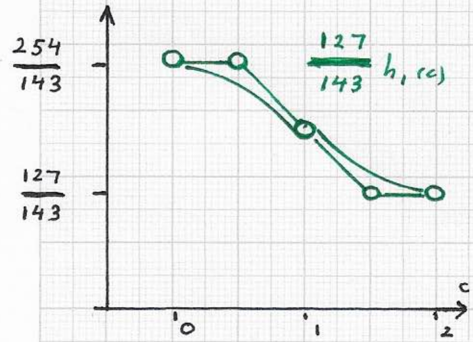
$$\Rightarrow \int_1^2 h_1(c)^2 = \frac{1}{5}\left(\frac{9}{4} + \frac{3}{2} + \frac{7}{6} + 1 + 1\right) = \frac{83}{60}$$

$$\Rightarrow \|h_1(c)\| = L_1 = \left(\frac{203}{60} + \frac{83}{60}\right)^{1/2} = \sqrt{\frac{143}{30}} \approx 2.18 \ \ldots$$



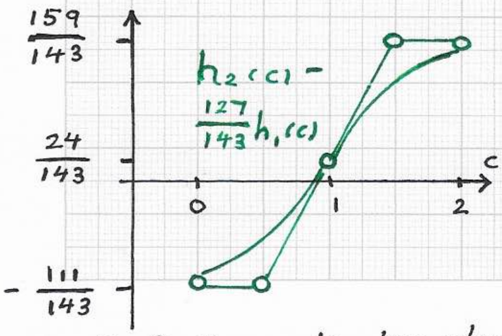
1. Normalization step:

$$\bar{h}_1(c) = h_1(c) / L_1 = \sqrt{\frac{30}{143}} \, h_1(c)$$



Projection of $h_2(c)$ onto $\bar{h}_1(c)$:

$$\langle h_2(c), \bar{h}_1(c) \rangle \bar{h}_1(c) = \frac{127}{\sqrt{30}\,\sqrt{143}} \bar{h}_1(c) = \frac{127}{143} h_1(c)$$

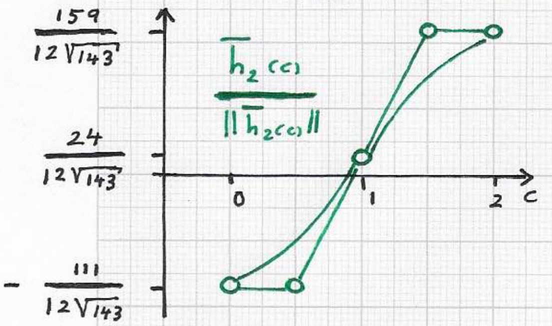


2. Orthogonalization step:

$$\bar{h}_2(c) = h_2(c) - \frac{127}{143} h_1(c)$$

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions: ... $\Rightarrow \overline{h}_1(c) = h_1(c)/L_1 = \sqrt{\frac{30}{143}}\, h_1(c)$



2. Orthonormalization of $h_2(c)$

$$\overline{h}_2(c) = h_2(c) - \langle h_2(c), \overline{h}_1(c)\rangle\, \overline{h}_1(c)$$

$$= h_2(c) - \underbrace{\langle h_2(c), h_1(c)\rangle \cdot \frac{1}{L_1}}_{=\ell_{2,1}} \cdot \overline{h}_1(c)$$

Normalization step:

$$\overline{h}_2(c) = \overline{h}_2(c)/L_2$$

$$= \left(h_2(c) - \frac{127}{143} h_1(c)\right)/L_2$$

$$= \frac{\sqrt{143}}{12}\left(h_2(c) - \frac{127}{143} h_1(c)\right)$$

$$\langle h_2(c), h_1(c)\rangle = \int_0^2 h_2(c)\, h_1(c) = \int_0^1 h_2(c)\, h_1(c) + \int_1^2 h_2(c)\, h_1(c)$$

$[0,1]$: $h_2(c)\, h_1(c) = 2B_0^4 + 2B_1^4 + \frac{25}{12}B_2^4 + \frac{9}{4}B_3^4 + \frac{9}{4}B_4^4$

$$\Rightarrow \int_0^1 h_2(c)\, h_1(c) = \frac{1}{5}\left(2 + 2 + \frac{25}{12} + \frac{9}{4} + \frac{9}{4}\right) = \frac{127}{60}$$

$[1,2]$: $h_2(c)\, h_1(c) = \frac{9}{4}B_0^4 + \frac{9}{4}B_1^4 + \frac{25}{12}B_2^4 + 2B_3^4 + 2B_4^4$

$$\Rightarrow \int_1^2 h_2(c)\, h_1(c) = \frac{1}{5}\left(\frac{9}{4} + \frac{9}{4} + \frac{25}{12} + 2 + 2\right) = \frac{127}{60}$$

$$\Rightarrow \langle h_2(c), h_1(c)\rangle = \frac{254}{60} = \frac{127}{30}$$

$$\Rightarrow \ell_{2,1} = \frac{127}{30} \cdot \sqrt{\frac{30}{143}} = \frac{127}{\sqrt{30}\sqrt{143}}$$

$$\Rightarrow \overline{h}_2(c) = h_2(c) - \frac{127}{\sqrt{30}\sqrt{143}}\, \overline{h}_1(c)$$

$$= h_2(c) - \frac{127}{\sqrt{30}\sqrt{143}} \cdot \sqrt{\frac{30}{143}}\, h_1(c)$$

$$= h_2(c) - \frac{127}{143}\, h_1(c)$$

Resulting upper triangular matrices needed for basis transformation – and coordinate/coefficient transformation:

$$T = \begin{bmatrix} L_1 & \ell_{2,1} \\ 0 & L_2 \end{bmatrix} = \begin{bmatrix} \sqrt{\frac{143}{30}} & \frac{127}{\sqrt{30}\sqrt{143}} \\ 0 & \frac{12}{\sqrt{143}} \end{bmatrix}$$

$$\Rightarrow T^{-1} = \begin{bmatrix} 1/L_1 & -\ell_{2,1}/(L_1 L_2) \\ 0 & 1/L_2 \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{\frac{30}{143}} & -\frac{127}{12\sqrt{143}} \\ 0 & \frac{\sqrt{143}}{12} \end{bmatrix}$$

Normalization of $\overline{h}_2(c)$

$$\|\overline{h}_2(c)\| = \left(\int_0^2 \overline{h}_2(c)^2\right)^{1/2} = \left(\int_0^1 \overline{h}_2(c) + \int_1^2 \overline{h}_2(c)\right)^{1/2}$$

$[0,1]$: $\overline{h}_2(c)^2 = \left(-\frac{111}{143}\right)^2 B_0^4 + \left(-\frac{111}{143}\right)^2 B_1^4 + \frac{2\left(-\frac{111}{143}\right)\left(\frac{24}{143}\right) + 4\left(-\frac{111}{143}\right)^2}{6} B_2^4 + \left(-\frac{111}{143}\right)\left(\frac{24}{143}\right) B_3^4 + \left(\frac{24}{143}\right)^2 B_4^4$

$$\Rightarrow \int_0^1 \overline{h}_2(c)^2 = \frac{1}{5}\left(\left(-\frac{111}{143}\right)^2 + \ldots + \left(\frac{24}{143}\right)^2\right)$$

$$= \frac{1992}{6833}$$

...

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions: … $[1,2]: \bar{h}_2(c)^2 = \left(\frac{24}{143}\right)^2 B_0^4 + \left(\frac{24}{143}\right)\left(\frac{159}{143}\right) B_1^4$

$$+ \frac{2\left(\frac{24}{143}\right)\left(\frac{159}{143}\right) + 4\left(\frac{159}{143}\right)}{6} B_2^4$$

$$+ \left(\frac{159}{143}\right)^2 B_3^4 + \left(\frac{159}{143}\right)^2 B_4^4$$

$$\Rightarrow \int_1^2 \bar{h}_2(c)^2 = \frac{1}{5}\left(\left(\frac{24}{143}\right)^2 + \ldots + \left(\frac{159}{143}\right)^2\right) = \frac{14616}{20499}$$

$$\Rightarrow \|\bar{h}_2(c)\| = \left(\frac{1992}{6833} + \frac{14616}{20499}\right)^{1/2} = \sqrt{\frac{144}{143}} = \frac{12}{\sqrt{143}} = L_2$$

$$\Rightarrow \bar{h}_2(c) = \bar{h}_2(c) / L_2 = \frac{\sqrt{143}}{12}\left(h_2(c) - \frac{127}{143} h_1(c)\right) = \frac{1}{12}\left(\sqrt{143}\, h_2(c) - \frac{127}{\sqrt{143}} h_1(c)\right)$$

3. Projection of $h(c)$ onto orthonormal basis vectors $\bar{h}_1(c)$ and $\bar{h}_2(c)$

$$\langle h(c), \bar{h}_1(c) \rangle = \int_0^2 h(c)\bar{h}_1(c) = \int_0^1 h(c)\bar{h}_1(c) + \int_1^2 h(c)\bar{h}_1(c)$$

$[0,1]: h(c) = 1,\ \bar{h}_1(c) = \sqrt{\frac{30}{143}}\, h_1(c) = \sqrt{\frac{30}{143}}\left(2B_0^2 + 2B_1^2 + \frac{3}{2}B_2^2\right)$

$$\Rightarrow \int_0^1 h(c)\bar{h}_1(c) = \sqrt{\frac{30}{143}} \cdot \frac{1}{3}\left(2 + 2 + \frac{3}{2}\right) = \frac{11}{6}\sqrt{\frac{30}{143}}$$

$[0,2]: h(c) = 1,\ \bar{h}_1(c) = \sqrt{\frac{30}{143}}\, h_1(c) = \sqrt{\frac{30}{143}}\left(\frac{3}{2}B_0^2 + 1B_1^2 + 1B_2^2\right)$

$$\Rightarrow \int_0^2 h(c)\bar{h}_1(c) = \sqrt{\frac{30}{143}} \cdot \frac{1}{3}\left(\frac{3}{2} + 1 + 1\right) = \frac{7}{6}\sqrt{\frac{30}{143}}$$

$$\Rightarrow \langle h(c), \bar{h}_1(c) \rangle = \frac{18}{6}\sqrt{\frac{30}{143}} = 3\sqrt{\frac{30}{143}}$$

$$\langle h(c), \bar{h}_2(c) \rangle = \int_0^2 h(c)\bar{h}_2(c) = \int_0^1 h(c)\bar{h}_2(c) + \int_1^2 h(c)\bar{h}_2(c)$$

$[0,1]: h(c) = 1,\ \bar{h}_2(c) = -\frac{111}{12\sqrt{143}} B_0^2 - \frac{111}{12\sqrt{143}} B_1^2 + \frac{24}{12\sqrt{143}} B_2^2$

$$\Rightarrow \int_0^1 h(c)\bar{h}_2(c) = \frac{1}{3} \cdot \frac{-198}{12\sqrt{143}} = -\frac{11}{2\sqrt{143}}$$

$[0,2]: h(c) = 1,\ \bar{h}_2(c) = \frac{24}{12\sqrt{143}} B_0^2 + \frac{159}{12\sqrt{143}} B_1^2 + \frac{159}{12\sqrt{143}} B_2^2$

$$\Rightarrow \int_1^2 h(c)\bar{h}_2(c) = \frac{1}{3} \cdot \frac{342}{12\sqrt{143}} = \frac{19}{2\sqrt{143}}$$

$$\Rightarrow \langle h(c), \bar{h}_2(c) \rangle = \frac{-11+19}{2\sqrt{143}} = \frac{4}{\sqrt{143}}$$

Resulting representation of $h(c)$ with respect to orthonormal basis:

$$h(c) = \langle h(c), \bar{h}_1(c) \rangle \cdot \bar{h}_1(c) + \langle h(c), \bar{h}_2(c) \rangle \cdot \bar{h}_2(c)$$

$$= 3\sqrt{\frac{30}{143}}\, \bar{h}_1(c) + \frac{4}{\sqrt{143}}\, \bar{h}_2(c)$$

Final representation of $h(c)$ with respect to basis $\{h_1(c), h_2(c)\}$:

$$h(c) = \begin{pmatrix} h_1(c) & h_2(c) \end{pmatrix} \cdot T^{-1} \cdot \begin{pmatrix} 3\sqrt{\frac{30}{143}} \\ \frac{4}{\sqrt{143}} \end{pmatrix}$$

$$= \begin{pmatrix} h_1(c) & h_2(c) \end{pmatrix} \begin{pmatrix} \sqrt{\frac{30}{143}} & -\frac{127}{12\sqrt{143}} \\ 0 & \frac{\sqrt{143}}{12} \end{pmatrix} \begin{pmatrix} 3\sqrt{\frac{30}{143}} \\ \frac{4}{\sqrt{143}} \end{pmatrix}$$

$$= \begin{pmatrix} h_1(c) & h_2(c) \end{pmatrix} \begin{pmatrix} 1/3 \\ 1/3 \end{pmatrix}$$

$$= \frac{1}{3} h_1(c) + \frac{1}{3} h_2(c)$$

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~ BH