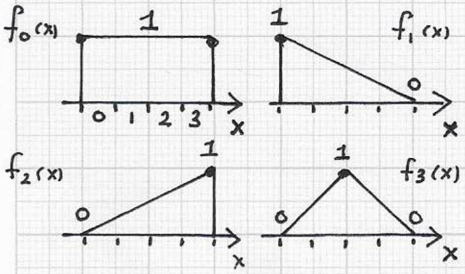
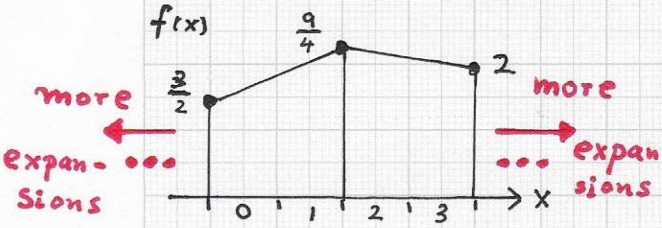
■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions:...

Hierarchical, multi-scale function approximation:



Certain basis functions, e.g., eigenfunctions, defined for a specific 4-element (4-lixel) domain used for approximation



Analysis of a given (local) function – a part of a segment – leads to the expansion of $f(x)$ in terms of the used basis functions, i.e.,

$$f(x) = \sum_{i=0}^{3} c_i f_i(x).$$

Here, the resulting coefficient values are $(c_0, c_1, c_2, c_3) = (1, 1/2, 1, 1/2)$. The figure indicates that this expansion is just one of several local function expansions. All local expansions together define coefficient value histograms for c_0-, c_1-, c_2-, c_3-values.

● In general, the number of "histogram sample functions" also depends on the sample segments stored for each class to be considered in a multi-class classification problem. Thus, the number of training samples used for "training" – k_1 for class 1, ..., k_C for the last class C – affects the indexing as follows:

$$\boxed{\begin{array}{l} {}^1h_1(c_1), \ldots, {}^nh_1(c_n), \\ \ldots \\ {}^1h_{k_1}(c_1), \ldots, {}^nh_{k_1}(c_n), \\ \vdots \\ {}^1h_{k_C}(c_1), \ldots, {}^nh_{k_C}(c_n) \end{array}}$$

are ALL stored "histogram sample functions" for ALL C classes.

● Given ${}^1h(c_1), \ldots, {}^nh(c_n)$, one can difference:

FOR all classes cl DO (cl=1... C)

  FOR all class segments sg DO (sg=1...k_{cl})

    FOR all coeff. value histograms i DO (i=1...n)

      → compute and store the "difference" between ${}^ih(c_i)$ and ${}^ih^{cl}_{sg}(c_i)$.

...

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• *Laplacian eigenfunctions:* ...

$d_{cl,sg,i}$ table

	1	2	3	scale
class 1	$d_{1,1,1}$ .8	.9	$d_{1,1,3}$ .8	2 segments
	.01	.02	.03	
class 2	.9	.8	.9	2 segments
	.7	.6	.01	
class 3	.01	.03	.9	2 segments
	.8 $d_{3,2,1}$	.8	.9 $d_{3,2,3}$	

$\varepsilon_{cl,sg,i}$ table

detection threshold values for all computed differences

.1	.1	.2
.1	.1	.1
.2	.2	.3
.1	.3	.1
.3	.2	.1
.1	.2	.2

detection table

"detection bits" set to + if $d_{cl,sg,i} \leq \varepsilon_{cl,sg,i}$ (set to − otherwise)

			D
−	−	−	−
+	+	+	+
−	−	−	−
−	−	+	?
+	+	−	?
−	−	−	−

Use of $d_{cl,sg,i}$ values for classification and detection. The column **D** in the detection table indicates an "all-scales match" (+), "some-scales match" (?) and "no-scales match" (−) with the new unclassified segment.

● Based on the chosen measure for "difference" one obtains distance values $d_{cl,sg,i}$ to be used for the classification of the unclassified segment with "histogram functions" $^1h(c_1), \ldots, {}^nh(c_n)$. We will refer to the index i, $i = 1 \ldots n$, also as **SCALE**. One can think of the $d_{cl,sg,i}$ values as a table organized as follows:

class 1	$d_{1,1,1}$	…	$d_{1,1,n}$	k_1 segments
	⋮			
	$d_{1,k_1,1}$	…	$d_{1,k_1,n}$	
class 2	$d_{2,1,1}$	…	$d_{2,1,n}$	k_2 segments
	⋮			
	$d_{2,k_2,1}$	…	$d_{2,k_2,n}$	
⋮				
class C	$d_{C,1,1}$	…	$d_{C,1,n}$	k_C segments
	⋮			
	$d_{C,k_C,1}$	…	$d_{C,k_C,n}$	

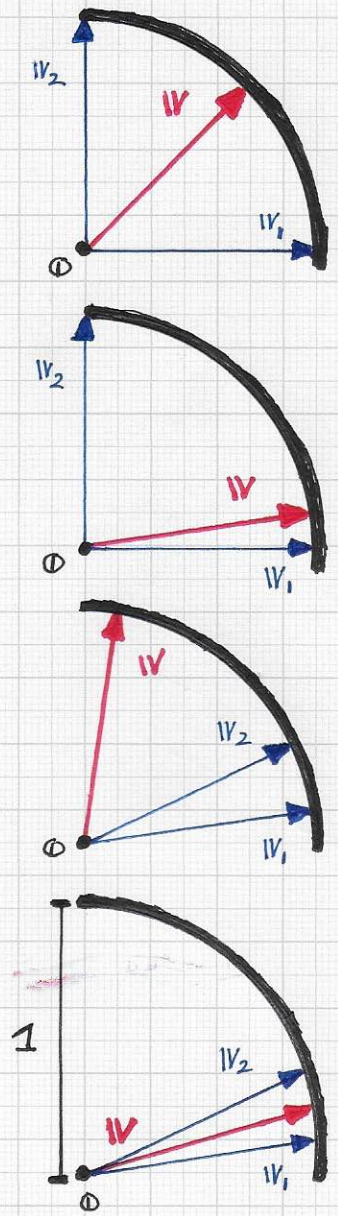
1 … n → scale

THIS TABLE MAKES POSSIBLE A MULTI-CLASS, MULTI-SEGMENT AND MULTI-SCALE CLASSIFICATION.

~ BH ...

OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions: … Defining and computing the $d_{cl,sg,i}$ values – "$d_{\text{class, segment, scale}}$" – is crucially important when comparing the coefficient "histogram function" ${}^{i}h(c_i)$ for the i^{th} scale with stored and classified "histogram sample functions" for several classes, with each class represented by multiple segments. The comparison of these functions ${}^{i}h_{sg}^{cl}(c_i)$ and the functions ${}^{i}h(c_i)$ must be based on a meaningful measure for "function difference" that one must also be able to calculate in a numerical stable and efficient way. For the derivation and better understanding of such a difference measure, we first consider the simple(r) case of defining differences for vectors in a geometrical setting, see left figure. More precisely, given a vector $\mathbf{v}$ (to be "classified") and its expansion as $\mathbf{v} = \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2$ in the basis $\{\mathbf{v}_1, \mathbf{v}_2\}$, where the basis vectors represent two "classes," determine the subsets of $\{\mathbf{v}_1, \mathbf{v}_2\}$ that have as their elements the basis vectors that belong to "$\mathbf{v}$'s class." …



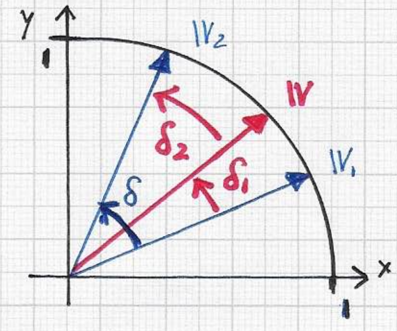
Geometrical setting. The vector $\mathbf{v}$ is represented via vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ as a linear combination $\mathbf{v} = \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2$. The goal is to use the coefficient values of α_1 and α_2 to define "difference" for $\mathbf{v}$ and $\mathbf{v}_1$, and $\mathbf{v}$ and $\mathbf{v}_2$.

OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

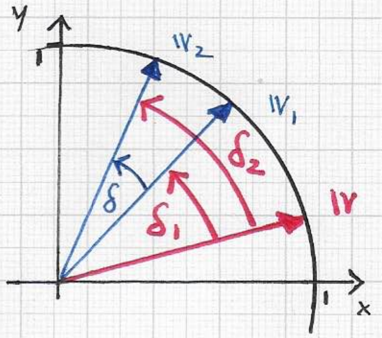
• Laplacian eigenfunctions: ... We assume that the vectors involved in the simple 2D setting illustrated on the previous page are all of length one. **Assumptions / conditions:**

i) The values of all coordinates of all vectors are non-negative.

ii) The angles between all pairs of vectors are positive / non-negative; "positive" is understood as counter-clockwise.



Two "classes" are defined by vectors $\mathbb{V}_1$ and $\mathbb{V}_2$. "Class distance" is to be calculated for an "unclassified vector" $\mathbb{V}$. The three angles indicated can be used to compute a "class distance."



The top figure shows an example where the "unclassified vector" $\mathbb{V}$ "lies between" the two "class vectors"; the bottom figure shows a case where the "unclassified vector" is not "between" the "class vectors." And:

$$\cos\delta = \langle \mathbb{V}_1, \mathbb{V}_2 \rangle,$$

$$\cos\delta_1 = \langle \mathbb{V}, \mathbb{V}_1 \rangle,$$

$$\cos\delta_2 = \langle \mathbb{V}, \mathbb{V}_2 \rangle,$$

$$\delta, \delta_1, \delta_2 \in [0, \pi].$$

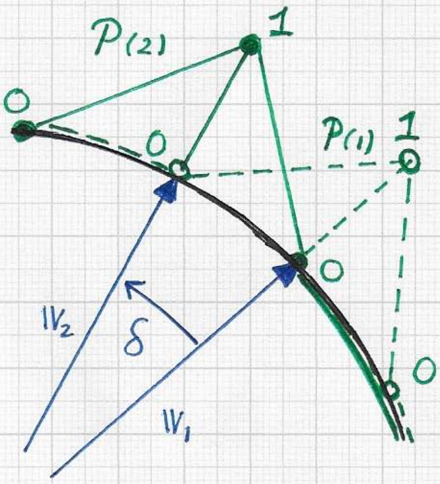
The goal is to define a "relative distance" for a vector $\mathbb{V}$ and "class vectors" $\mathbb{V}_1$ and $\mathbb{V}_2$. The term "relative" is used as the distance between $\mathbb{V}_1$ and $\mathbb{V}_2$ can be used to establish the distance **1** and other distances can be measured relatively to this unit distance. For example, we can use "angle distance" to define a "class probability" for $\mathbb{V}$. Such a probability should be **1** when $\mathbb{V}$ is equal to a class vector and decrease linearly to **0** with increasing "angle distance" — **RELATIVELY** to δ:

$$P(i) = \max\{0, (\delta - \delta_i)/\delta\}, \quad \delta \neq 0,$$

is a "Linear B-spline-like" function that satisfies the desired properties of probability $P(i)$, $i \in \{1, 2\}$. ...

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

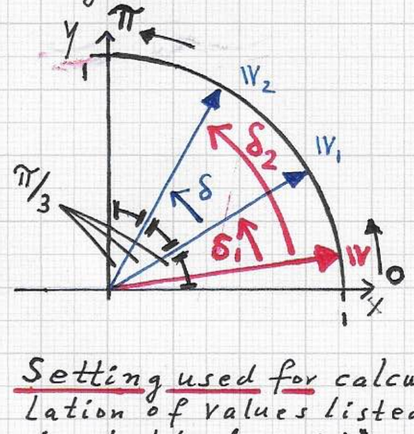
• Laplacian eigenfunctions: ... In other words, the probability $P(i)$ of an "unclassified vector" $\mathbb{V}$ belonging to the i^{th} "vector class" has a linearly varying value between **0** and **1**, where the range of δ_i-values leading to a positive value of $P(i)$ is defined by δ. The figure (left) shows the graphs of two functions $P(1)$ and $P(2)$ for the 2D setting. We provide an example for the computation of these functions (see figure, left, bottom):



Principle underlying class probability functions $P(1)$ and $P(2)$. These functions have their maximal value 1 when an "unclassified vector" is equal to a "class vector," and they reach their minimal value 0 when "angle distance" to a "class vector" is equal to or larger than δ.

δ	δ_1	δ_2	$P(1)$	$P(2)$
$\pi/3$	$\pi/3$	$2\pi/3$	0	0
$\pi/3$	0	$\pi/3$	1	0
$\pi/3$	$\pi/3$	0	0	1
$\pi/3$	$2\pi/3$	$\pi/3$	0	0

$$P(i) = \max\{0, (\delta - \delta_i)/\delta\}$$



Setting used for calculation of values listed in table (right).

• **Note.** One can generalize this definition of the probability function $P(i)$ in two simple ways (of practical relevance):

i) Instead of using a linear decrease of $P(i)$ from **1** to 0, one can use a more general polynomial decrease.

ii) Instead of using **2·δ** as the domain region with associated positive P-values, one can scale this value, using a smaller region.

...