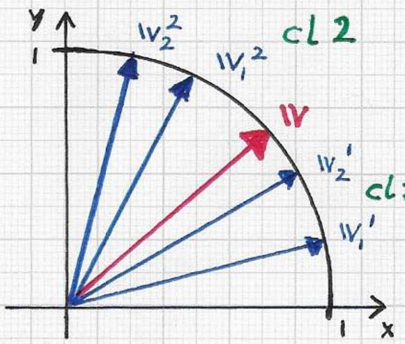
■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions: ...



Geometrically equivalent setting of multi-class, multi-segment data: Vectors $\mathbb{V}_1^1$, $\mathbb{V}_2^1$ represent class 1, segments 1 and 2; vectors $\mathbb{V}_1^2$, $\mathbb{V}_2^2$ represent class 2, segments 1 and 2. In this setting, one must determine whether a vector $\mathbb{V}$ is "close enough" to a vector $\mathbb{V}_{sg}^{cl}$ to be interpreted as a representative of class cl (cl = class, sg = segment); cl, sg $\in \{1, 2\}$.

The two expansions of $\mathbb{V}$ are already known at this time:

$$\mathbb{V} = \alpha_1^1 \mathbb{V}_1^1 + \alpha_2^1 \mathbb{V}_2^1,$$

$$\mathbb{V} = \alpha_1^2 \mathbb{V}_1^2 + \alpha_2^2 \mathbb{V}_2^2.$$

(The aspect of multi-scale data processing must also be included in the material classification calculations.)

• Note. In the decision-making context, i.e., when a vector $\mathbb{V}$ must be classified, one must use class-specific thresholds ε_i to determine whether $\mathbb{V}$ is of class i or not:

"Vector $\mathbb{V}$ is of class i
$:\Leftrightarrow \; P(i) \geq \varepsilon_i$."

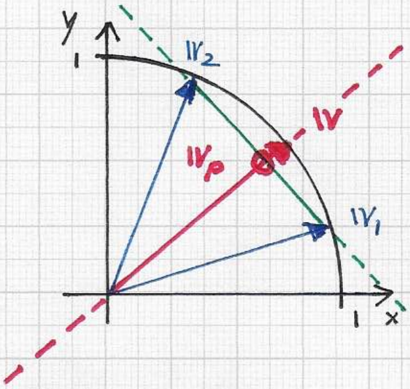
We have viewed the vectors $\mathbb{V}_1$ and $\mathbb{V}_2$ as "class vectors" in this geometrical setting. In our driving problem of material classification, we have k_{cl} class-specific sements to be considered. We can also "transfer" the concept of segments to the geometrical setting: For example, two classes, each represented by two segments, transfer to a geometrical setting where $\mathbb{V}_1^1$ and $\mathbb{V}_2^1$ are the two vectors of the first class ($k_1 = 2$) and $\mathbb{V}_1^2$ and $\mathbb{V}_2^2$ are the two vectors of the second class ($k_2 = 2$). The figure (Left) illustrates this example. The more general classification condition becomes:

"Vector $\mathbb{V}$ is of class cl
$:\Leftrightarrow \; P(cl, sg) \geq \varepsilon_{sg}^{cl}$ for at least one $sg \in \{1, k_{cl}\}$."

...

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions: ... Another closely related concept that one can employ for determining how close a vector/point is to a set of "basis vectors/points" is the concept of BARYCENTRIC COORDINATES. In the geometrical setting of our driving classification problem, the question to be answered is the following: "Given a set of linearly independent 'class vectors/points' – or 'segment vectors/points defining a class – how can a barycentric-coordinate-based representation of an unclassified vector/point be used for classification?" First, one must explore how barycentric coordinates apply to the geometrical setting and how one can compute them. In a simple example (left figure), we consider the case of two vectors $\mathbb{V}_1$ and $\mathbb{V}_2$ to be used to represent a vector $\mathbb{V}$; again all vectors are of unit length and have only non-negative coordinate values. IT MUST BE EMPHASIZED THAT THE COMPUTED COORDINATES ARE NOT BASED ON EXACT HYPER-SPHERICAL GEOMETRY TO REPRESENT $\mathbb{V}$ WITHOUT ERROR; THEY MERELY SERVE AS CLASSIFICATION FEATURES. ...



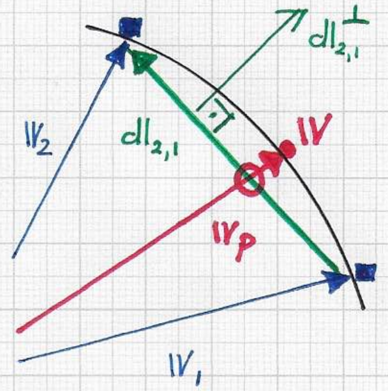
Geometrical construction involved in calculating a representation of $\mathbb{V}$ via barycentric coordinates. The coordinates are NOT defined relative to the circular arc on which all vectors end, rather they are obtained by linearizing the problem by considering the line passing through the points where the vectors $\mathbb{V}_1$ and $\mathbb{V}_2$ end:

ONE INTERSECTS THE LINE $\alpha \cdot \mathbb{V}$ WITH THE LINE THROUGH THE "END POINTS" OF $\mathbb{V}_1$ AND $\mathbb{V}_2$, PRODUCING $\mathbb{V}_p$. THIS PROJECTED POINT CAN BE REPRESENTED EXACTLY WITH BARYCENTRIC COORDINATES RELATIVE TO $\mathbb{V}_1, \mathbb{V}_2$. THESE COORDINATES CAN SUBSEQUENTLY ALSO BE USED TO CHARACTERIZE $\mathbb{V}$'S "RELATIONSHIP" TO $\mathbb{V}_1$ AND $\mathbb{V}_2$.

OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions: ... In the following, the term "vector" is sometimes used when the term "point" is the geometrically proper term (and vice versa). We must compute the point $\mathbb{V}_p$ obtained by intersecting the parametrically defined line $\mathbb{C}(t) = t \cdot \mathbb{V}$ with the implicitly defined line through points $\mathbb{V}_1$ and $\mathbb{V}_2$, i.e., $Ax + By + C = 0$. One inserts $\mathbb{C}(t) = (t x_0, t y_0)^T$ into the implicit line and obtains $t A x_0 + t B y_0 + C = 0$, or $t = -C / (A x_0 + B y_0)$. The point $\mathbb{V}_1$ lies on the implicit line; thus, $C = -A x_1 - B y_1$. Further, the coefficients A and B are given by the (outward) normal vector $\mathbb{dl}_{2,1}^{\perp}$ as $A = d_y$ and $B = -d_x$. One obtains $t = (A x_1 - B y_1) / (A x_0 + B y_0) = \frac{d_y x_1 - d_x y_1}{d_y x_0 - d_x y_0} = \langle \mathbb{dl}_{2,1}^{\perp}, \mathbb{V}_1 \rangle / \langle \mathbb{dl}_{2,1}^{\perp}, \mathbb{V} \rangle$. Thus, the point $\mathbb{V}_p$ is given as

$$\boxed{\mathbb{V}_p = \frac{\langle \mathbb{dl}_{2,1}^{\perp}, \mathbb{V}_1 \rangle}{\langle \mathbb{dl}_{2,1}^{\perp}, \mathbb{V} \rangle} \, \mathbb{V}} \;.$$



The vectors $\mathbb{V}$, $\mathbb{V}_1$ and $\mathbb{V}_2$ are the positional vectors of the points shown as ● and ■. These vectors are therefore also considered as points. Relevant quantities in this figure are:

$$\mathbb{V} = (x_0, y_0)^T$$
$$\mathbb{V}_1 = (x_1, y_1)^T$$
$$\mathbb{V}_2 = (x_2, y_2)^T$$
$$\mathbb{V}_p = (\overline{x}_0, \overline{y}_0)^T$$
$$\mathbb{dl}_{2,1} = \mathbb{V}_2 - \mathbb{V}_1 = (d_x, d_y)^T = (x_2 - x_1, y_2 - y_1)^T$$
$$\mathbb{dl}_{2,1}^{\perp} = (d_y, -d_x)^T$$

(= outward normal vector)

The point $\mathbb{V}_p$ can now be represented with the use of barycentric coordinates relative to $\mathbb{V}_1$ and $\mathbb{V}_2$ as

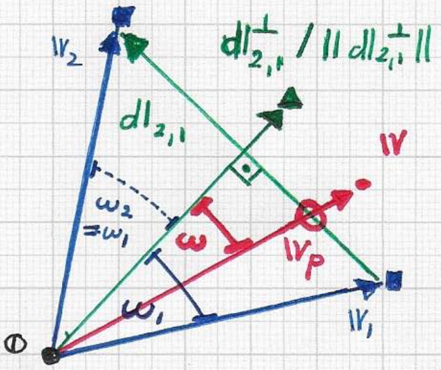
$$\mathbb{V}_p = u_1 \mathbb{V}_1 + u_2 \mathbb{V}_2 \;, \quad u_1 + u_2 = 1 \;.$$

The barycentric representation of $\mathbb{V}_p$, the tuple (u_1, u_2) defines a "feature tuple" that can be used for the classification of $\mathbb{V}_p$.

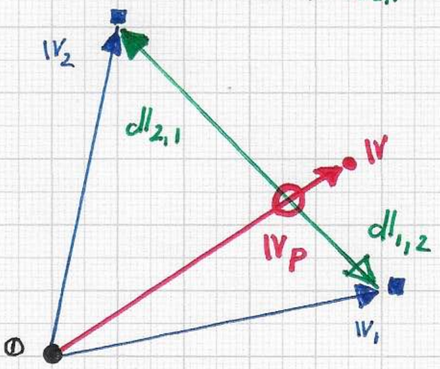
...

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

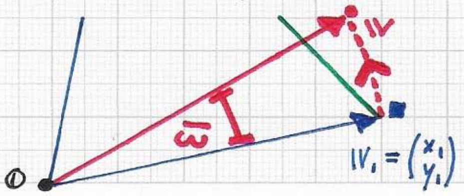
• Laplacian eigenfunctions: ...



All vectors/points shown are normalized.* One can express $\mathbb{v}_p$ purely in terms of angles ω and ω_1.
(*except $\mathbb{d}_{2,1}$)



Using vectors/positional vectors to write $\mathbb{v}_p$ via vector addition and as "weighted combination" of points via barycentric coordinates for $\mathbb{v}_1$ and $\mathbb{v}_2$. Change in orientation is illustrated by showing both $\mathbb{d}_{2,1} = \mathbb{v}_2 - \mathbb{v}_1$ and $\mathbb{d}_{1,2} = \mathbb{v}_1 - \mathbb{v}_2$.



Understanding $\mathbb{v}$ as the result of rotating $\mathbb{v}_1$.

• Note. The illustration (left) shows that $\cos\omega_1 = \langle \mathbb{d}_{2,1}^{\perp} / \|\mathbb{d}_{2,1}^{\perp}\|, \mathbb{v}_1 \rangle$ and $\cos\omega = \langle \mathbb{d}_{2,1}^{\perp} / \|\mathbb{d}_{2,1}^{\perp}\|, \mathbb{v} \rangle$. Thus, considering that $\|\mathbb{d}_{2,1}\| = \|\mathbb{d}_{2,1}^{\perp}\|$, one can also write $\mathbb{v}_p$ as

$$\mathbb{v}_p = (\cos\omega_1 / \cos\omega)\, \mathbb{v} .$$

• Note. The second illustration (left, middle) shows the relationship between a "vector-based representation" and a "point-based (barycentric) representation" of $\mathbb{v}_p$. One can write $\mathbb{v}_p$ as follows:

$$\begin{aligned} \mathbb{v}_p &= \mathbb{v}_1 + u\, \mathbb{d}_{2,1} \\ &= \mathbb{v}_1 + u\, (\mathbb{v}_2 - \mathbb{v}_1) \\ &= (1-u)\, \mathbb{v}_1 + u\, \mathbb{v}_2 \quad , u \in [0,1] \\ &= \mathbb{v}_2 + \bar{u}\, \mathbb{d}_{1,2} \\ &= \mathbb{v}_2 + \bar{u}\, (\mathbb{v}_1 - \mathbb{v}_2) \\ &= (1-\bar{u})\, \mathbb{v}_2 + \bar{u}\, \mathbb{v}_1 \\ &\overset{\bar{u}=(1-u)}{=} u\, \mathbb{v}_2 + (1-u)\, \mathbb{v}_1 \end{aligned}$$

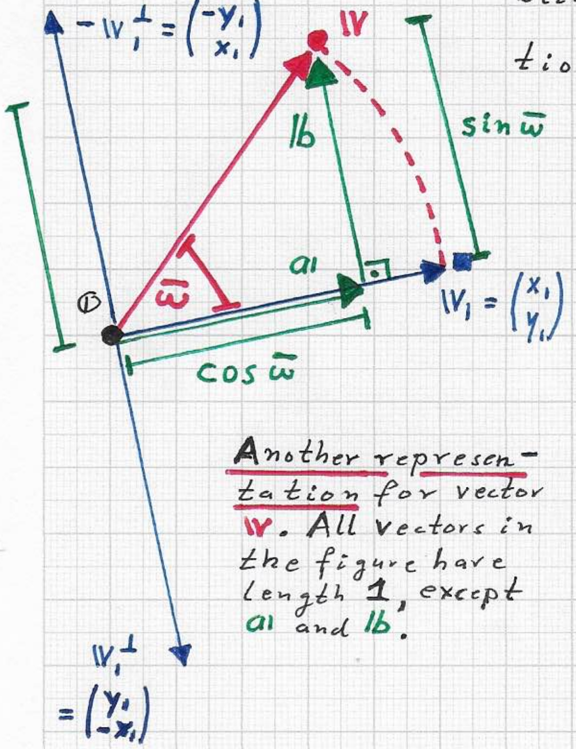
• Note. The movement of $\mathbb{v}$ on the unit circle can also be viewed as a rotation and be expressed via a rotation matrix:

$$\mathbb{v} = Rot(\bar{\omega})\, \mathbb{v}_1 = \begin{pmatrix} \cos\bar{\omega} & -\sin\bar{\omega} \\ \sin\bar{\omega} & \cos\bar{\omega} \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix},$$

$\bar{\omega} \in [0, 2\omega_1]$ ...

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions: ... • **Note.** There exists yet another possibility to write the unit vector/positional vector $\mathbf{w}$ (left figure):

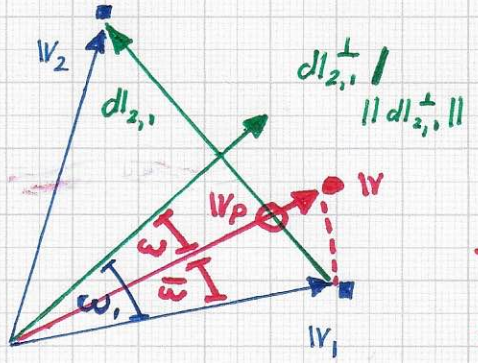


Another representation for vector $\mathbf{w}$. All vectors in the figure have length 1, except $\mathbf{a}$ and $\mathbf{b}$.

$$\mathbf{w} = \mathbf{a} + \mathbf{b}$$
$$= \cos(\bar{\omega})\,\mathbf{w}_1 - \sin(\bar{\omega})\,\mathbf{w}_1^{\perp}$$
$$= \langle \mathbf{w}, \mathbf{w}_1 \rangle\,\mathbf{w}_1 - \langle \mathbf{w}, -\mathbf{w}_1^{\perp} \rangle\,\mathbf{w}_1^{\perp}$$
$$= \langle \mathbf{w}, \mathbf{w}_1 \rangle\,\mathbf{w}_1 + \langle \mathbf{w}, \mathbf{w}_1^{\perp} \rangle\,\mathbf{w}_1^{\perp}.$$

• **Note.** We can also express $\mathbf{w}_p$ in a slightly different way (left figure, bottom):

$$\mathbf{w}_p = \frac{\cos\omega_1}{\cos\omega}\,Rot(\bar{\omega})\,\mathbf{w}_1$$
$$= \frac{\cos\omega_1}{\cos\omega}\begin{pmatrix} \cos\bar{\omega} & -\sin\bar{\omega} \\ \sin\bar{\omega} & \cos\bar{\omega} \end{pmatrix}\mathbf{w}_1$$
$$= \frac{\langle \mathbf{d}_{2,1}^{\perp}, \mathbf{w}_1 \rangle}{\langle \mathbf{d}_{2,1}^{\perp}, \mathbf{w} \rangle}\begin{pmatrix} \langle \mathbf{w}, \mathbf{w}_1 \rangle & \langle \mathbf{w}, \mathbf{w}_1^{\perp} \rangle \\ -\langle \mathbf{w}, \mathbf{w}_1^{\perp} \rangle & \langle \mathbf{w}, \mathbf{w}_1 \rangle \end{pmatrix}\mathbf{w}_1.$$



Vectors and angles necessary to express $\mathbf{w}_p$ purely in terms of scalar products.

• **Here:** The "embedding space" is **2-dim.**; we "project" $\mathbf{w}$ onto the **1-dim.** line through $\mathbf{w}_1$ and $\mathbf{w}_2$; $\mathbf{w}_p$ is the "**0-dim.** point" that results.

The use and advantage of these representations based on SCALAR PRODUCTS of vectors is important for the general, high-dimensional setting. For example, when $\mathbf{w}$, $\mathbf{w}_1$, $\mathbf{w}_2$ and $\mathbf{w}_3$ are unit vectors in three-dimensional space, $\mathbf{w}_p$ is a point in the plane defined by $\mathbf{w}_1$, $\mathbf{w}_2$ and $\mathbf{w}_3$...