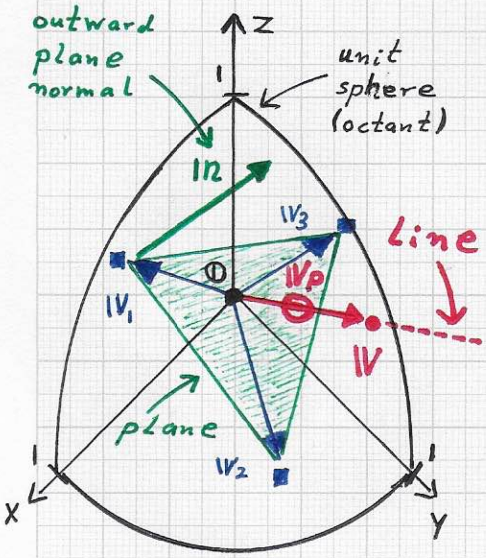
■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions: ... Before generalizing these ideas to the general multi-dimensional case, we discuss the generalization from the unit circle to the unit sphere. In this case, we must compute the point $\mathbb{V}_p$ in the (hyper-)plane passing through points $\mathbb{V}_1$, $\mathbb{V}_2$ and $\mathbb{V}_3$. We represent the line through the origin with direction $\mathbb{V}$ in parametric form as $\mathbb{C}(t) = t\,\mathbb{V} = t\,(x_0, y_0, z_0)^T$. The plane is written in implicit form as $Ax + By + Cz + D = 0$. Inserting the terms from the line equation into the plane equation. First, we establish the plane equation: The plane's (outward) normal vector $\mathbb{n}$ is given as $\mathbb{n} = \mathbb{d}_{2,1} \times \mathbb{d}_{3,1}$, where $\mathbb{d}_{2,1} = (x_2 - x_1, y_2 - y_1, z_2 - z_1)^T$ and $\mathbb{d}_{3,1} = (x_3 - x_1, y_3 - y_1, z_3 - z_1)^T$. We denote the components of this normal, the result of the cross product, as $\mathbb{n} = (n_x, n_y, n_z)^T$. This vector defines the coefficient vector of the plane equation, in part: $(A, B, C)^T = (n_x, n_y, n_z)^T$. Since $\mathbb{V}_1$ is a specific point in the plane, it follows that $D = -(Ax_1 + By_1 + Cz_1)$.

...



3D "embedding space." The vectors $\mathbb{V}$, $\mathbb{V}_1$, $\mathbb{V}_2$ and $\mathbb{V}_3$ are normalized positional vectors of the indicated associated points. The line through the origin $\mathbb{O}$ with direction $\mathbb{V}$ intersects the plane through $\mathbb{V}_1$, $\mathbb{V}_2$ and $\mathbb{V}_3$ in the point $\mathbb{V}_p$. All vector/point coordinate values are non-negative.

The point $\mathbb{V}_p$ must be determined. It is computed by calculating the intersection of the parametrically defined line with the implicitly defined plane.

$$\mathbb{V} = (x_0, y_0, z_0)^T$$

$$\mathbb{V}_i = (x_i, y_i, z_i)^T$$

$$\mathbb{V}_p = (\bar{x}_0, \bar{y}_0, \bar{z}_0)^T$$

$$\mathbb{d}_{j,1} = \mathbb{V}_j - \mathbb{V}_1 = (d_x^{j,1}, d_y^{j,1}, d_z^{j,1})^T$$

OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions: ... Thus, the plane equation is given as

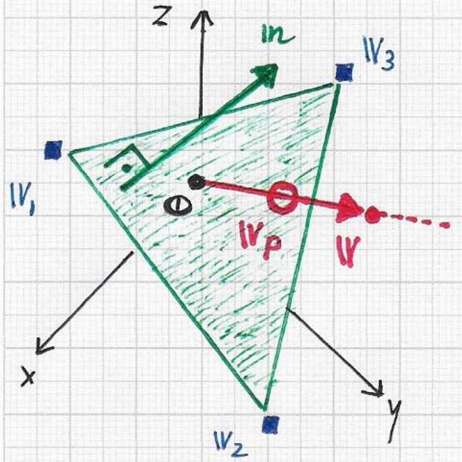
$$x n_x + y n_y + z n_z = x_1 n_x + y_1 n_y + z_1 n_z .$$

Inserting the expressions for the line yields

$$t x_0 n_x + t y_0 n_y + t z_0 n_z = x_1 n_x + y_1 n_y + z_1 n_z .$$

Solving this equation for t leads to

$$t = \frac{\langle \mathbb{v}_1 , \mathbb{n} \rangle}{\langle \mathbb{v} , \mathbb{n} \rangle} .$$



Essential parameters defining the point $\mathbb{v}_p$ in the plane through $\mathbb{v}_1$, $\mathbb{v}_2$ and $\mathbb{v}_3$.

Second, we compute the needed point $\mathbb{v}_p$ in the plane by using this specific t-value in the line equation:

$$\mathbb{v}_p = \frac{\langle \mathbb{v}_1 , \mathbb{n} \rangle}{\langle \mathbb{v} , \mathbb{n} \rangle} \mathbb{v} .$$

⇒ General case: The (1D) line defined by $\overrightarrow{\mathbb{0}\mathbb{v}}$ is intersected in k-dim. space with the (k-1)-dim. hyper-plane that passes through the k points with positional vectors $\mathbb{v}_1$, ..., $\mathbb{v}_k$.

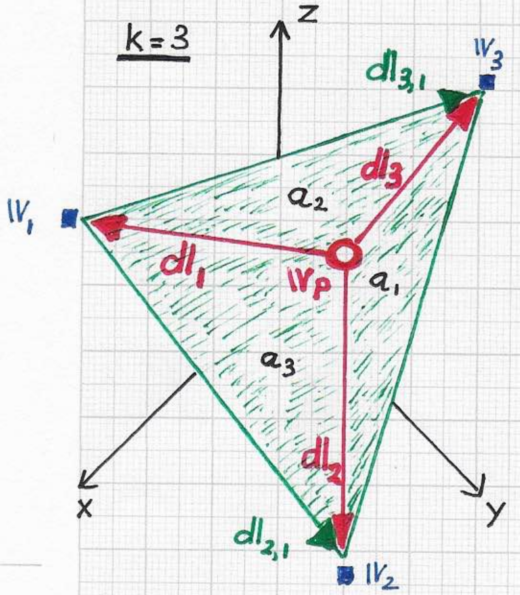
Comparing this equation to the one on page 18 (2/11/2022) makes it clear that "projecting $\mathbb{v}$" onto a line and onto a plane are – algebraically – equal.

The goal is the computation of "good" approximations of $\mathbb{v}$'s SPHERICAL BARYCENTRIC COORDINATES (ON THE UNIT SPHERE) relative to the spherical triange with corner vertices with positional vectors $\mathbb{v}_1$, $\mathbb{v}_2$ and $\mathbb{v}_3$.

...

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions: ... INSTEAD OF DEFINING AND COMPUTING (HYPER-) SPHERICAL BARYCENTRIC COORDINATES FOR $\mathbb{V}$ – WHICH WOULD BE BASED ON RATIOS OF (HYPER-) SPHERICAL PATCH "AREAS" – WE EXPRESS THE POINT $\mathbb{V}_p$ WITH STANDARD BARYCENTRIC COORDINATES DEFINED BY THE (k-1)-DIMENSIONAL SIMPLEX WITH VERTICES $\mathbb{V}_1, \mathbb{V}_2, \ldots, \mathbb{V}_{k-1}$ THAT IS EMBEDDED IN k-DIMENSIONAL SPACE.



Barycentric coordinate computation for k = 3. The points $\mathbb{V}_1$, $\mathbb{V}_2$ and $\mathbb{V}_3$ define the simplex (triangle) used to calculate the barycentric coordinates of $\mathbb{V}_p$. These four points lie in the same (hyper-)plane, but $\mathbb{V}_p$ can lie inside or outside the simplex (triangle).

The point $\mathbb{V}_p$ splits the triangle with corner points $\mathbb{V}_1$, $\mathbb{V}_2$ and $\mathbb{V}_3$ into three sub-triangles obtained by connecting $\mathbb{V}_p$ to the corner points. The resulting sub-triangle areas – SIGNED AREAS – are a_1, a_2 and a_3. Thus, the total area of the triangle is $a_1+a_2+a_3$.

$\mathbb{V}_i = (x_i, y_i, z_i)^T$

$\mathbb{V}_p = (\bar{x}_0, \bar{y}_0, \bar{z}_0)^T$

(The vertices $\mathbb{V}_1, \mathbb{V}_2, \ldots, \mathbb{V}_{k-1}$ can also be viewed as (k-1) linearly independent positional vectors of the corner points of the (k-1)-dimsional simplex that is defined in k-dimensional space.)

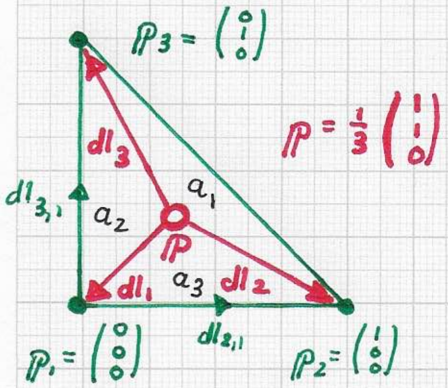
We consider the case $k=3$ in detail (left figure) and define and compute $\mathbb{V}_p$'s barycentric coordinates relative to the triangle with vertices/corner points $\mathbb{V}_1$, $\mathbb{V}_2$ and $\mathbb{V}_3$. We must compute the quantities constituting the representation

$$\mathbb{V}_p = \sum_{i=1}^{3} u_i \mathbb{V}_i = \sum_{i=1}^{3} \frac{a_i}{a_1+a_2+a_3} \mathbb{V}_i \quad .$$

...

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions: ... We summarize the main steps involved in the computation of the barycentric coordinates u_i, $i=1\ldots3$.



Barycentric coordinates of $\mathbb{P}$ – $\mathbb{P}$ "inside":

$\mathbb{d}_{2,1} = (1,0,0)^T$

$\mathbb{d}_{3,1} = (0,1,0)^T$

$\mathbb{d}_1 = (-1/3,-1/3,0)^T$

$\mathbb{d}_2 = (2/3,-1/3,0)^T$

$\mathbb{d}_3 = (-1/3,2/3,0)^T$

$\Rightarrow\ \mathbb{n} = (0,0,1)^T$

$\mathbb{n}_1 = (0,0,1/3)^T$

$\mathbb{n}_2 = (0,0,1/3)^T$

$\mathbb{n}_3 = (0,0,1/3)^T$

$\Rightarrow\ A = |A| = \frac{1}{2}$

$a_i = |a_i| = \frac{1}{6}$

$\Rightarrow\ u_i = a_i / A = \frac{1}{3}$

$\Rightarrow\ \mathbb{P} = \frac{1}{3}\mathbb{P}_1 + \frac{1}{3}\mathbb{P}_2 + \frac{1}{3}\mathbb{P}_3$

First, we define the (difference) vectors

$$\mathbb{d}_{j,1} = \mathbb{v}_j - \mathbb{v}_1 \quad , j = 2,3$$

$$\mathbb{d}_i = \mathbb{v}_i - \mathbb{v}_p \quad , i = 1,2,3$$

Second, the SIGNED areas are given by the lengths of certain cross products (normal vectors) of these (difference) vectors as follows:

• Oriented triangle normal:

$$\mathbb{n} = \mathbb{d}_{2,1} \times \mathbb{d}_{3,1}$$

• Oriented sub-triangle normals:

$$\mathbb{n}_1 = \mathbb{d}_2 \times \mathbb{d}_3 , \quad \mathbb{n}_2 = \mathbb{d}_3 \times \mathbb{d}_1 , \quad \mathbb{n}_3 = \mathbb{d}_1 \times \mathbb{d}_2 .$$

Ignoring for the moment that **ORIENTATION AND SIGNS must be considered**, the absolute values of triangle areas are defined by the lengths of these normals:

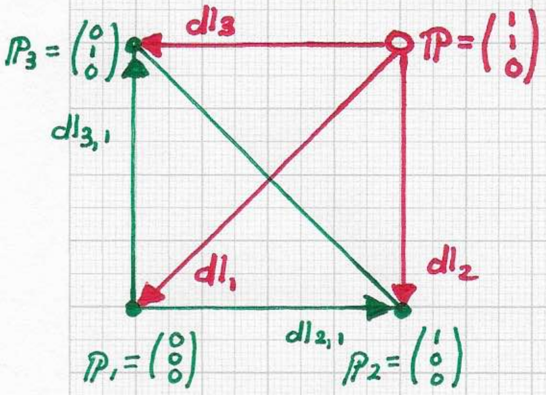
$$|A| = \frac{1}{2}\|\mathbb{n}\|, \quad |a_i| = \frac{1}{2}\|\mathbb{n}_i\| .$$

• **Note.** The barycentric coordinates satisfy the equation $\sum_{i=1}^{k} u_i = 1$. A point $\mathbb{P} = \sum_{i=1}^{k} u_i \mathbb{P}_i$ is a **convex combination** when $u_i \geq 0$, $i=1\ldots k$. In this case, $\mathbb{P}$ lies in the interior or on the boundary of the simplex with vertices $\mathbb{P}_i$, $i=1\ldots k$.

...

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• **Laplacian eigenfunctions:** ... We review the use of determinants for the computation of **oriented normal vectors** (**outward and inward normal vectors**), since these vectors define the **signed values of $\mathbb{P}$'s barycentric coordinates**. The use of determinants will be relevant for generalizing the concepts to the $(k-1)$-dimensional simplex embedded in k-dimensional space and the "non-geometrical setting." We consider the specific example shown in the figure (left):



Barycentric coordinates of $\mathbb{P}$ – $\mathbb{P}$ "outside":

$dl_{2,1} = (1,0,0)^T$

$dl_{3,1} = (0,1,0)^T$

$dl_1 = (-1,-1,0)^T$

$dl_2 = (0,-1,0)^T$

$dl_3 = (-1,0,0)^T$

$\Rightarrow \ \mathbf{n} = (0,0,1)^T$

$\mathbf{n}_1 = (0,0,\underline{\underline{-1}})^T$

$\mathbf{n}_2 = (0,0,1)^T$

$\mathbf{n}_3 = (0,0,1)^T$

$\Rightarrow \ A = \frac{1}{2}$

$a_1 = \underline{\underline{-\frac{1}{2}}}$

$a_2 = a_3 = \frac{1}{2}$

$\Rightarrow \ u_1 = \underline{\underline{-1}}, \ u_2 = u_3 = 1$

$\Rightarrow \ \underline{\mathbb{P} = -1\mathbb{P}_1 + 1\mathbb{P}_2 + 1\mathbb{P}_3}$

$$\mathbf{n}: \begin{vmatrix} | & | & X \\ dl_{2,1} & dl_{3,1} & Y \\ | & | & Z \end{vmatrix} = \begin{vmatrix} 1 & 0 & X \\ 0 & 1 & Y \\ 0 & 0 & Z \end{vmatrix} = 1 \cdot Z$$

$$\Rightarrow \underline{\mathbf{n} = (0,0,1)^T}$$

$$\mathbf{n}_1: \begin{vmatrix} | & | & X \\ dl_2 & dl_3 & Y \\ | & | & Z \end{vmatrix} = \begin{vmatrix} 0 & -1 & X \\ -1 & 0 & Y \\ 0 & 0 & Z \end{vmatrix} = \underline{\underline{-1}} \cdot Z$$

$$\Rightarrow \underline{\mathbf{n}_1 = (0,0,\underline{\underline{-1}})^T}$$

$$\mathbf{n}_2: \begin{vmatrix} | & | & X \\ dl_3 & dl_1 & Y \\ | & | & Z \end{vmatrix} = \begin{vmatrix} -1 & -1 & X \\ 0 & -1 & Y \\ 0 & 0 & Z \end{vmatrix} = 1 \cdot Z$$

$$\Rightarrow \underline{\mathbf{n}_2 = (0,0,1)^T}$$

$$\mathbf{n}_3: \begin{vmatrix} | & | & X \\ dl_1 & dl_2 & Y \\ | & | & Z \end{vmatrix} = \begin{vmatrix} -1 & 0 & X \\ -1 & -1 & Y \\ 0 & 0 & Z \end{vmatrix} = 1 \cdot Z \Rightarrow \underline{\mathbf{n}_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}}$$

...