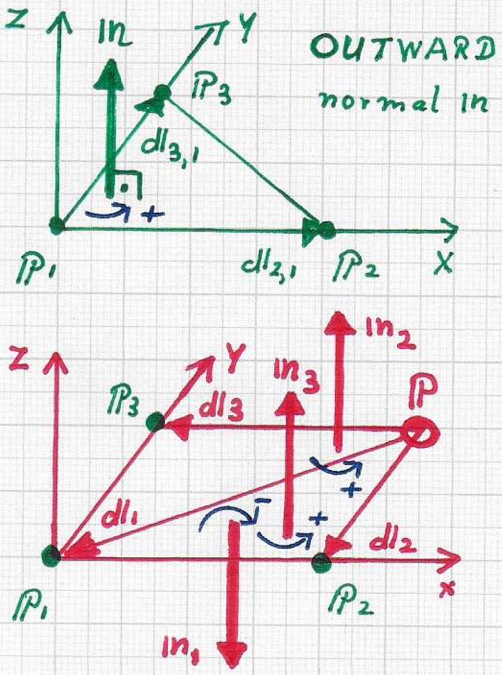
Bernd Hamann

Stratovan

2/18/2022

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions: ...



Importance of orientation and indexing of points and vectors. Top: Definition of outward normal of triangle in 3D space. Bottom: Normals of three sub-triangles defined by "split" point $\mathbb{p}$; two normals have outward and one normal has inward direction.

The triangle shown in the top figure is the "reference triangle." Its vertices $\mathbb{p}_1$, $\mathbb{p}_2$ and $\mathbb{p}_3$ and the associated difference vectors $d\mathbb{l}_{2,1}$ and $d\mathbb{l}_{3,1}$ determine the cross product to be computed, generating outward normal $\mathbb{n}$.

THE SIGNED AREA RATIOS a_i / A DEFINE $\mathbb{p}$'S BARYCENTRIC COORDINATES.

Two facts must be considered when computing the barycentric coordinates of a point $\mathbb{p}$ that lies in the same plane as the triangle with oriented vertices $\mathbb{p}_1$, $\mathbb{p}_2$ and $\mathbb{p}_3$:

i) The areas of the triangle and the three sub-triangles have the ABSOLUTE values $\frac{1}{2!}\|\mathbb{n}\|$ and $\frac{1}{2!}\|\mathbb{n}_i\|$, $i = 1 \ldots 3$.

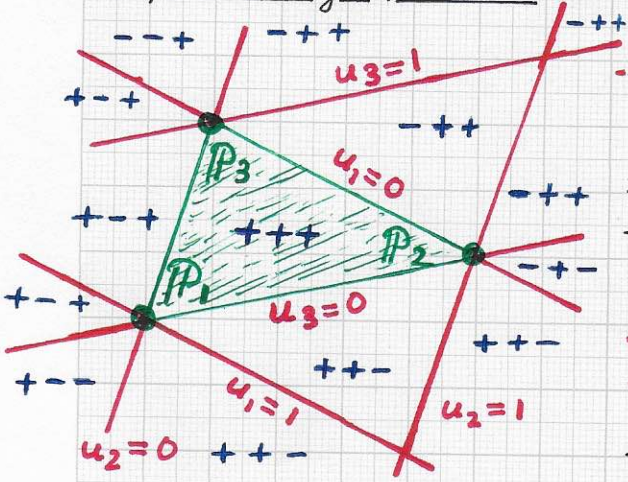
ii) The normal $\mathbb{n}$ of the "reference triangle" defines the the triangle's (and the plane's) OUTWARD direction – "pointing into the positive half-space"; thus, the SIGN of $\frac{1}{2!}\|\mathbb{n}\|$ is $+$ and the triangle's signed area is $A = +\frac{1}{2!}\|\mathbb{n}\|$. Based on the position of $\mathbb{p}$, sub-triangle normals have either OUTWARD or INWARD direction. Thus, **THE SIGN OF $\langle \mathbb{n}, \mathbb{n}_i \rangle$ DEFINES THE SIGN OF $\frac{1}{2!}\|\mathbb{n}_i\|$: $a_i = \mathrm{SIGN}(\langle \mathbb{n}, \mathbb{n}_i \rangle)\frac{1}{2!}\|\mathbb{n}_i\|$.**

(The case $\mathbb{n}_i = \mathbb{0}$ implies that $a_i = 0$.)

...

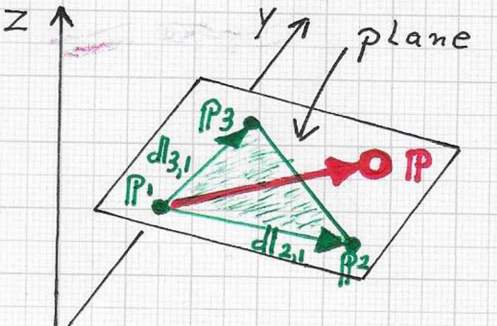
■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions: ...



Iso-lines $u_i=0$ and $u_i=1$ partitioning the plane into triangular regions, relative to a "reference triangle" with vertices $\mathbb{P}_1$, $\mathbb{P}_2$ and $\mathbb{P}_3$.

Points in the interior of grid triangles have barycentric coordinates with triangle-specific "sign triples" $s_1 s_2 s_3$.



Plane containing "reference triangle" and point $\mathbb{P}$ to be expressed in barycentric coordinates in a 3D embedding space. $\mathbb{P}=(x_0,y_0,z_0)^T$.

The figure (left) shows that a plane is partitioned into triangles that are identical to the "reference triangle" defined by $\mathbb{P}_1$, $\mathbb{P}_2$ and $\mathbb{P}_3$. The barycentric coordinate iso-lines $u_i = \ldots, -2, -1, 0, 1, 2, \ldots$, $i=1,2,3$, define the grid lines shown in the figure. One can associate a "sign triple" $s_1 s_2 s_3$, $s_i \in \{+,-\}$, with points that lie in the interior of the grid triangles. Points lying inside the "reference triangle" have the "sign triple" $+++$. The figure shows triples for several triangles.

It is also important to recall the relationships between analytical geometry, linear algebra, linear equation systems and determinants (e.g., Cramer's rule) – and their relevance for computing barycentric coordinates in the general multi-dimensional case. Considering the example shown in the figure (left, bottom), $\mathbb{P}$'s barycentric representation is

$$\mathbb{P} = (1-u_2-u_3)\,\mathbb{P}_1 + u_2\,\mathbb{P}_2 + u_3\,\mathbb{P}_3 \ . \quad \ldots$$

OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions: ... The unknown coordinates u_2 and u_3 must be computed; the value of u_1 is defined by the equation $u_1 = 1 - u_2 - u_3$. One has to solve the following linear equation system:

$$(1-u_2-u_3)\,\mathbb{P}_1 + u_2\,\mathbb{P}_2 + u_3\,\mathbb{P}_3 = \mathbb{P}$$

$$(1-u_2-u_3)\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + u_2 \begin{pmatrix} x_2 \\ y_2 \\ z_0 \end{pmatrix} + u_3 \begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}$$

$$\begin{pmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \\ z_2 - z_1 & z_3 - z_1 \end{pmatrix} \begin{pmatrix} u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} x_0 - x_1 \\ y_0 - y_1 \\ z_0 - z_1 \end{pmatrix}$$

$$\begin{pmatrix} | & | \\ \mathbf{d}_{2,1} & \mathbf{d}_{3,1} \\ | & | \end{pmatrix} \begin{pmatrix} u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} | \\ \mathbf{d}_{0,1} \\ | \end{pmatrix}$$

$$D \cdot \mathbf{u} = \mathbf{d}_{0,1}$$

Least squares: $D^T D \cdot \mathbf{u} = D^T \mathbf{d}_{0,1}$

$$\begin{pmatrix} \langle \mathbf{d}_{2,1}, \mathbf{d}_{2,1} \rangle & \langle \mathbf{d}_{2,1}, \mathbf{d}_{3,1} \rangle \\ \langle \mathbf{d}_{3,1}, \mathbf{d}_{2,1} \rangle & \langle \mathbf{d}_{3,1}, \mathbf{d}_{3,1} \rangle \end{pmatrix} \cdot \mathbf{u} = \begin{pmatrix} \langle \mathbf{d}_{0,1}, \mathbf{d}_{2,1} \rangle \\ \langle \mathbf{d}_{0,1}, \mathbf{d}_{3,1} \rangle \end{pmatrix}$$

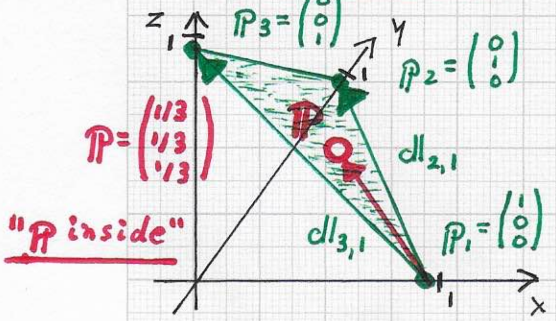
!

$$\Rightarrow \mathbf{u} = \left(D^T D\right)^{-1} D^T \mathbf{d}_{0,1}.$$

Since the number of unknown coordinates is 2 (u_2, u_3) and the dimension of the embedding space is 3, one can compute the solution via least squares. It is interesting to see that all matrix and right-hand side entries are scalar products of vectors involving $\mathbb{P}_1$, $\mathbb{P}_2$, $\mathbb{P}_3$ and $\mathbb{P}$.

...

Simple example:



$\mathbf{d}_{2,1} = (-1, 1, 0)^T$

$\mathbf{d}_{3,1} = (-1, 0, 1)^T$

$\mathbf{d}_{0,1} = \mathbb{P} - \mathbb{P}_1 = \left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right)^T$

$\Rightarrow$ least-squares system:

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$\Rightarrow \quad u_2, u_3 = \frac{1}{3}$

$u_1 = 1 - u_2 - u_3 = \frac{1}{3}$

$\Rightarrow \mathbb{P} = \frac{1}{3}\mathbb{P}_1 + \frac{1}{3}\mathbb{P}_2 + \frac{1}{3}\mathbb{P}_3$

"$\mathbb{P}$ outside": e.g.,

$\mathbb{P} = (1, 1, -1)^T$

$\Rightarrow \mathbf{d}_{0,1} = (0, 1, -1)^T$

$\Rightarrow$ least-squares system:

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$\Rightarrow \mathbb{P} = 1\mathbb{P}_1 + 1\mathbb{P}_2 - 1\mathbb{P}_3$

Bernd Hamann

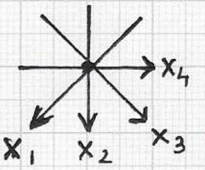
Stratovan

2/20/2022

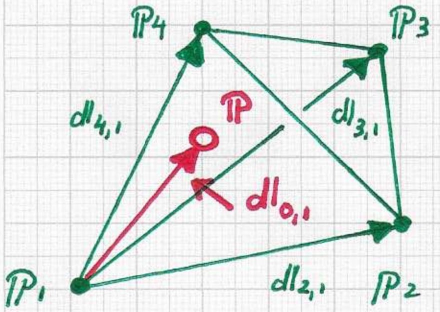
■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions: ... It is instructive to consider the case of a four-dimensional embedding space. The figure (left) illustrates the "projected" geometrical setting. In this case, all points/vectors involved have four coordinates – x_1, x_2, x_3 and x_4 – and the point $\mathbb{P}$ lies in the three-dimensional sub-space that is defined by the four points $\mathbb{P}_1, \mathbb{P}_2, \mathbb{P}_3$ and $\mathbb{P}_4$; these four points are the vertices of the simplex (tetrahedron) in 4D embedding space that is the "reference tetrahedron." The point $\mathbb{P}$ has barycentric coordinates relative to this tetrahedron. Thus, $\mathbb{P}$'s representation is

$$\mathbb{P} = (1-u_2-u_3-u_4)\,\mathbb{P}_1 + u_2\mathbb{P}_2 + u_3\mathbb{P}_3 + u_4\mathbb{P}_4 .$$



"projected" coordinate system of 4D embedding space



Representing point $\mathbb{P}$ in barycentric form with respect to a "reference tetrahedron" in 4D space.

Notation:

$\mathbb{P} = (x_1^0, x_2^0, x_3^0, x_4^0)^T$

$\mathbb{P}_i = (x_1^i, x_2^i, x_3^i, x_4^i)^T$

$dl_{i,1} = \mathbb{P}_i - \mathbb{P}_1$, $i=2 \ldots 4$

$dl_{0,1} = \mathbb{P} - \mathbb{P}_1$

Employing a purely algebraic generalization, the unknown barycentric coordinates result from

$$\begin{bmatrix} x_1^2 - x_1^1 & x_1^3 - x_1^1 & x_1^4 - x_1^1 \\ x_2^2 - x_2^1 & x_2^3 - x_2^1 & x_2^4 - x_2^1 \\ x_3^2 - x_3^1 & x_3^3 - x_3^1 & x_3^4 - x_3^1 \\ x_4^2 - x_4^1 & x_4^3 - x_4^1 & x_4^4 - x_4^1 \end{bmatrix} \begin{bmatrix} u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} x_1^0 - x_1^1 \\ x_2^0 - x_2^1 \\ x_3^0 - x_3^1 \\ x_4^0 - x_4^1 \end{bmatrix}, \text{ i.e.,}$$

$$\begin{bmatrix} dl_{2,1} & dl_{3,1} & dl_{4,1} \end{bmatrix} \begin{bmatrix} \mathbb{u} \end{bmatrix} = \begin{bmatrix} dl_{0,1} \end{bmatrix} .$$

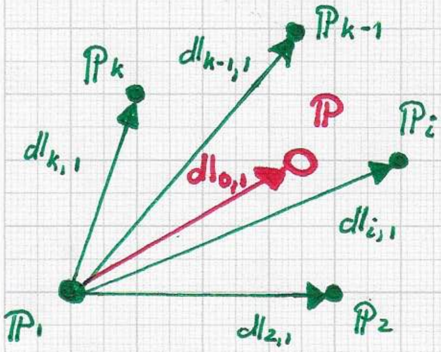
"The 4D embedding space contains the 3D simplex (tetrahedron) with 4 vertices, used to express $\mathbb{P}$ relative to the simplex."

⇒ **SOLVE SYSTEM $\mathbb{D} \cdot \mathbb{u} = dl_{0,1}$ VIA LEAST SQUARES!**

$$\mathbb{D} \cdot \mathbb{u} = dl_{0,1} \quad \ldots$$

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions: ... **Now** it is straightforward to generalize calculations to the multi-dimensional case:



"Projection" of data defining the barycentric coordinates of $\mathbb{P}$, relative to a "reference simplex" with **k** vertices.

$d\mathbb{I}_{i,1} = \mathbb{P}_i - \mathbb{P}_1 \ , i = 2 \dots k$

$d\mathbb{I}_{0,1} = \mathbb{P} - \mathbb{P}_1$

- → Points / vectors have k coordinates; the embedding space is k-dimensional.
- → Point $\mathbb{P}$ must be represented relative to a **(k-1)**-dimensional "reference simplex" with **k** vertices, $\mathbb{P}_1, \dots, \mathbb{P}_k$.
- → **The** representation of $\mathbb{P} = (x_1^0, \dots, x_k^0)^T$ is

$$\mathbb{P} = (1 - u_2 - \dots - u_k)\,\mathbb{P}_1 + \sum_{i=2}^{k} u_i \mathbb{P}_i \ .$$

- → Using matrix notation, the linear equation system for $\mathbb{P}$'s coordinates is

$$\begin{bmatrix} | & & | \\ d\mathbb{I}_{2,1} & \dots & d\mathbb{I}_{k,1} \\ | & & | \end{bmatrix} \begin{bmatrix} | \\ \mathbb{u} \\ | \end{bmatrix} = \begin{bmatrix} | \\ d\mathbb{I}_{0,1} \\ | \end{bmatrix} .$$

$$\mathbb{D} \cdot \mathbb{u} = d\mathbb{I}_{0,1}$$

- → **Solve** the overdetermined system via least squares:

$$\mathbb{u} = (\mathbb{D}^T \mathbb{D})^{-1} \mathbb{D}^T d\mathbb{I}_{0,1} \ ,$$ where

$$\begin{bmatrix} \langle d\mathbb{I}_{2,1}, d\mathbb{I}_{2,1} \rangle & \dots & \langle d\mathbb{I}_{2,1}, d\mathbb{I}_{k,1} \rangle \\ \vdots & & \vdots \\ \langle d\mathbb{I}_{k,1}, d\mathbb{I}_{2,1} \rangle & \dots & \langle d\mathbb{I}_{k,1}, d\mathbb{I}_{k,1} \rangle \end{bmatrix} \mathbb{u} = \begin{bmatrix} \langle d\mathbb{I}_{0,1}, d\mathbb{I}_{2,1} \rangle \\ \vdots \\ \langle d\mathbb{I}_{0,1}, d\mathbb{I}_{k,1} \rangle \end{bmatrix} .$$

• **The least squares solution defines the values for $u_2, \dots, u_k$, and $u_1 = 1 - \sum_{i=2}^{k} u_i$. The barycentric coordinate tuple $(u_1, \dots, u_k)$ can be used to determine how close, "how similar," $\mathbb{P}$ is to the points $\mathbb{P}_1, \dots, \mathbb{P}_k$.**

...