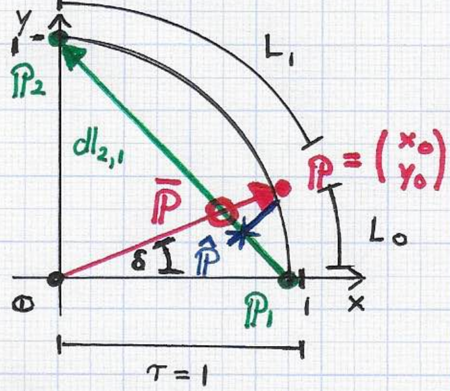
## OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions:




Three possibilities to define barycentric coordinates – for $\mathbb{P}$, $\overline{\mathbb{P}}$ and $\hat{\mathbb{P}}$. Specific values chosen:

$\mathbb{P} = (x_0, y_0)^T$,
$x_0^2 + y_0^2 = 1$,
$x_0, y_0 \geq 0$

$\overline{\mathbb{P}} = (\overline{x}_0, \overline{y}_0)^T$,
intersection of segments $\overline{\mathbb{O}\mathbb{P}}$ and $\overline{\mathbb{P}_1\mathbb{P}_2}$

$\hat{\mathbb{P}}$ = perpendicular projection of $\mathbb{P}$ onto $\overline{\mathbb{P}_1\mathbb{P}_2}$

$\delta = \measuredangle(\vec{\mathbb{P}}_1, \vec{\mathbb{P}})$,
$\cos\delta = x_0$

$\mathbb{d}_{2,1} = \mathbb{P}_2 - \mathbb{P}_1$
$= (-1, 1)^T$

$L_0 + L_1 = \frac{\pi}{2}$,
circle arc lengths

• **Note.** It is important to remember that the barycentric coordinates discussed on the preceding pages are merely approximations of the true "hyper-spherical barycentric coordinates" of a point $\mathbb{P}$ that lies on a unit hypersphere together with "reference points" $\mathbb{P}_1, \ldots, \mathbb{P}_k$. The figure (left) illustrates a specific setting for $k=2$ and three possible ways to associate barycentric coordinates with $\mathbb{P}$:

i.) Computing exact "hyper-spherical barycentric coordinates", i.e., lengths of circular arcs for $k = 2$ serve as basis:

$$L_0 = \delta \cdot r = \delta$$
$$L_1 = \frac{\pi}{2} - \delta = \frac{\pi}{2} - L_0$$

Apply normalization:

$$u = \frac{2}{\pi}\delta .$$

ii.) Representing $\overline{\mathbb{P}}$ in barycentric coordinates relative to simplex $\overline{\mathbb{P}_1\mathbb{P}_2}$:

• parametric line: $\mathbb{c}(t) = t \cdot \mathbb{P}$

• implicit line through $\mathbb{P}_1$ and $\mathbb{P}_2$:

$$x + y - 1 = 0$$

• intersection: $t x_0 + t y_0 = 1 \Rightarrow t = \frac{1}{x_0 + y_0}$

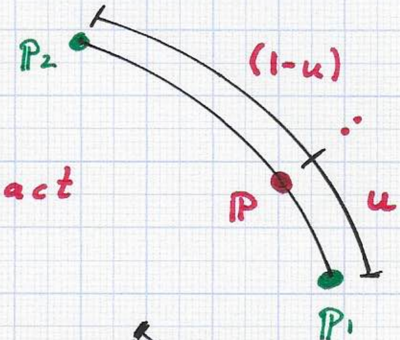
$$\Rightarrow \overline{\mathbb{P}} = (\overline{x}_0, \overline{y}_0)^T = \frac{1}{x_0 + y_0}(x_0, y_0)^T.$$

...

## ■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions: ... $\Rightarrow \overline{\mathbb{P}} - \mathbb{P}_1 = \overline{\mathbb{P}} - (1,0)^T = \frac{1}{x_0+y_0}(-y_0, y_0)^T = \overline{\mathrm{dl}}_{0,1}$

i.) Exact





Using the described least-squares method for computing $u$:

$$\left(\mathrm{dl}_{2,1}\right) u = \left(\overline{\mathrm{dl}}_{0,1}\right)$$

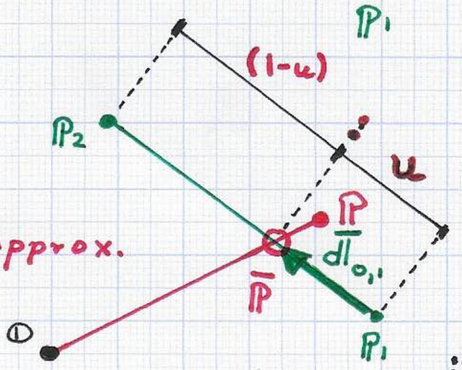
$$\Rightarrow \mathrm{dl}_{2,1}^T\, \mathrm{dl}_{2,1}\, u = \mathrm{dl}_{2,1}^T\, \overline{\mathrm{dl}}_{0,1}$$

$$(-1,1)\begin{pmatrix}-1\\1\end{pmatrix} u = (-1,1)\frac{1}{x_0+y_0}\begin{pmatrix}-y_0\\y_0\end{pmatrix}$$

$$2u = 2\frac{y_0}{x_0+y_0}$$

$$u = \frac{y_0}{x_0+y_0}\,.$$

ii.) Approx.





iii.) Representing $\mathbb{P}$ directly relative to simplex $\overline{\mathbb{P}_1\mathbb{P}_2}$ via a least-squares solution:

• $\mathrm{dl}_{0,1} = \mathbb{P} - \mathbb{P}_1 = (x_0 - 1, y_0)^T$

$$\left(\mathrm{dl}_{2,1}\right) u = \left(\mathrm{dl}_{0,1}\right)$$

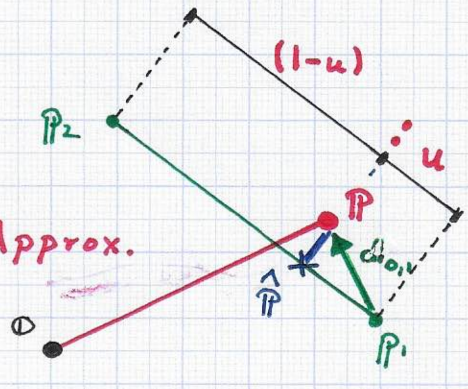
$$\Rightarrow \mathrm{dl}_{2,1}^T\, \mathrm{dl}_{2,1}\, u = \mathrm{dl}_{2,1}^T\, \mathrm{dl}_{0,1}$$

$$(-1,1)\begin{pmatrix}-1\\1\end{pmatrix} u = (-1,1)\begin{pmatrix}x_0-1\\y_0\end{pmatrix}$$

$$2u = 1 - x_0 + y_0$$

$$u = \frac{1-x_0+y_0}{2}\,.$$

iii.) Approx.





Barycentric coordinate computation/approximation of point $\mathbb{P}$. The second and third illustrations show the two described approximation methods for $u$.

The table (right, bottom) lists u-values obtained by the three methods for five chosen $\delta$-values.

Denoting the u-values as $u_{i.}$, $u_{ii.}$ and $u_{iii.}$, on obtains these values for five specific $\delta$:

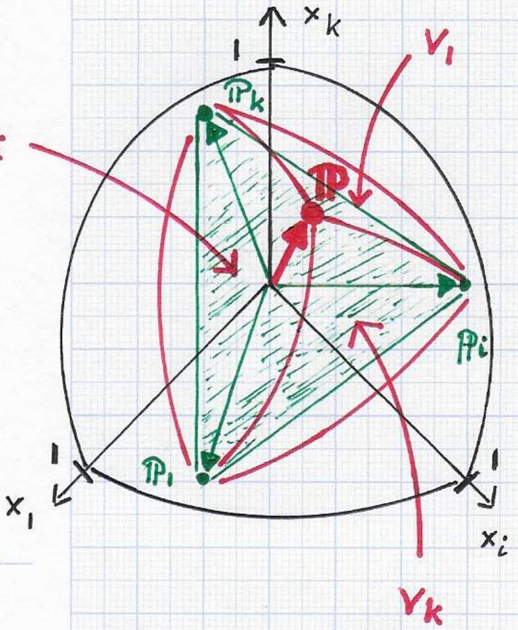
| | 0 | $\pi/8$ | $\pi/4$ | $3\pi/8$ | $\pi/2$ |
|---|---|---|---|---|---|
| $u_{i.}$ | 0 | 1/4 | 1/2 | 3/4 | 1 |
| $u_{ii.}$ | 0 | .293 | 1/2 | .707 | 1 |
| $u_{iii.}$ | 0 | .229 | 1/2 | .771 | 1 |

...

## ▪ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions: ... This table shows that the error value for $u$ when computed via ii.) is $\left|\frac{1}{4} - .293\right| = \left|\frac{3}{4} - .707\right|$ for $\delta = \frac{\pi}{8}$ and $\delta = \frac{3\pi}{8}$; when using iii.) to compute $u$-values, the error value for these two $\delta$-values is $\left|\frac{1}{4} - .229\right| = \left|\frac{3}{4} - .771\right|$. Thus, for these two $\delta$-values method ii.) leads to the error value .043 while method iii.) has the error value .021. Of course, one can perform a more detailed analysis of this error behavior — for a general, continuous and multi-dimensional error function — but this is beyond the needs of our driving application. Thus, we simply make the following recommendation:

**Given the point/positional vector $\mathbb{P}$ and $k$ points/positional vectors $\mathbb{P}_1, \ldots, \mathbb{P}_k$ defining a $(k-1)$-dimensional "reference simplex" for $\mathbb{P}$ — where $\mathbb{P}$ and $\mathbb{P}_1, \ldots, \mathbb{P}_k$ represent normalized, unit positional vectors — approximate $\mathbb{P}$'s hyper-spherical barycentric coordinate tuple by...**




Points/positional vectors $\mathbb{P}, \mathbb{P}_1, \ldots, \mathbb{P}_k$ lie on/end on a unit hyper-sphere in $k$-dimensional space. The exact, "true" hyper-spherical barycentric coordinates of $\mathbb{P}$ are defined by the hyper-spherical volumes (or areas) $V_1, \ldots, V_k$. For example, $\mathbb{P}$'s $i$-th "hyper-spherical barycentric coordinate is

$$u_i = \frac{V_i}{\sum_{j=1}^{k} V_j}.$$

Instead of calculating these coordinate values exactly — which is complex and computationally expensive — a least-squares approximation can be used.

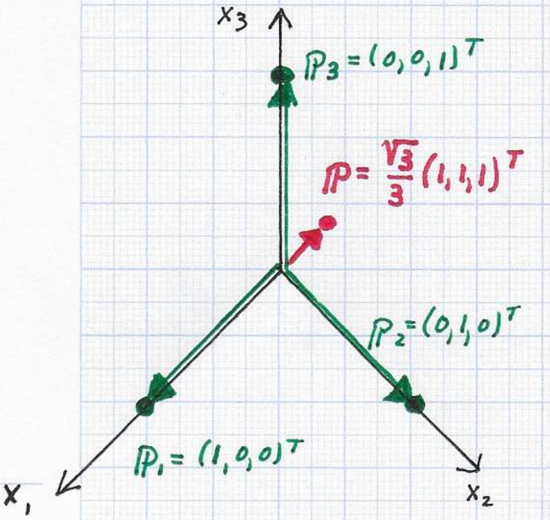
## OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions: ... **computing the coordinate values as $u_1 = 1 - \sum_{i=2}^{k} u_i$, $u_2, \ldots, u_k$ resulting from the least-squares solution of the over-determined system** !

$$\underbrace{\begin{bmatrix} | & & | \\ \mathbb{P}_2 - \mathbb{P}_1 & \cdots & \mathbb{P}_k - \mathbb{P}_1 \\ | & & | \end{bmatrix}}_{(k-1)\text{ columns}} \cdot \begin{bmatrix} u_2 \\ \vdots \\ u_k \end{bmatrix} = \begin{bmatrix} | \\ \mathbb{P} - \mathbb{P}_1 \\ | \end{bmatrix} \Big\} k \text{ rows}.$$

$$\mathbb{D} \quad \cdot \quad \mathbb{u} \quad = \quad \mathbb{d}_{0,1}.$$

Example: $k=3$




$\mathbb{d}_{2,1} = \mathbb{P}_2 - \mathbb{P}_1 = (-1,1,0)^T$

$\mathbb{d}_{3,1} = \mathbb{P}_3 - \mathbb{P}_1 = (-1,0,1)^T$

$\mathbb{d}_{0,1} = \mathbb{P} - \mathbb{P}_1 = \left(\frac{\sqrt{3}}{3} - 1, \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right)^T$

$\Rightarrow$ normal equations:

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$\Rightarrow$ $(u_2, u_3) = (1/3, 1/3)$

$\Rightarrow$ $u_1 = 1 - 2/3 = 1/3$

$\Rightarrow$ **The approximated "hyper-spherical barycentric coordinate" tuple of $\mathbb{P}$ is $(1/3, 1/3, 1/3)$.**

The difference vectors defining $\mathbb{D}$ are $\mathbb{d}_{i,1} = \mathbb{P}_i - \mathbb{P}_1$, $i = 2 \ldots k$, and the normal equations for the least-squares solution for $\mathbb{u}$ are given by

$$\begin{bmatrix} \langle \mathbb{d}_{2,1}, \mathbb{d}_{2,1} \rangle & \ldots & \langle \mathbb{d}_{2,1}, \mathbb{d}_{k,1} \rangle \\ \vdots & & \vdots \\ \langle \mathbb{d}_{k,1}, \mathbb{d}_{2,1} \rangle & \ldots & \langle \mathbb{d}_{k,1}, \mathbb{d}_{k,1} \rangle \end{bmatrix} \begin{bmatrix} u_2 \\ \vdots \\ u_k \end{bmatrix} = \begin{bmatrix} \langle \mathbb{d}_{0,1}, \mathbb{d}_{2,1} \rangle \\ \vdots \\ \langle \mathbb{d}_{0,1}, \mathbb{d}_{k,1} \rangle \end{bmatrix}.$$

$$\mathbb{D}^T \mathbb{D} \quad \cdot \quad \mathbb{u} \quad = \mathbb{D}^T \mathbb{d}_{0,1}.$$

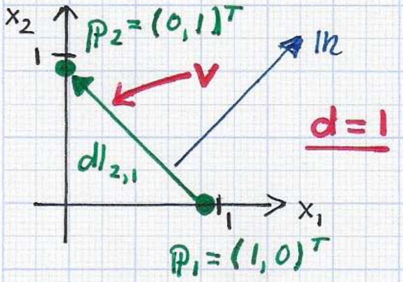
$$u_1 = 1 - u_2 - u_3 - \ldots - u_k.$$

• **Note.** In the context of classification, all the unit/normalized points/positional vectors represent histograms of materials/objects; $\mathbb{P}$'s associated $\mathbb{u}$-tuple is used to decide whether $\mathbb{P}$ is "close to" $\mathbb{P}_1$ or ... or $\mathbb{P}_k$. ...

## ▪ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions:

Simplex hyper-volumes for different dimensions (**d**):

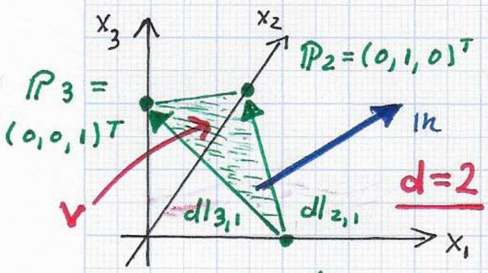



$d_{2,1} = (-1, 1)^T$

$\Rightarrow D = \begin{vmatrix} -1 & x_1 \\ 1 & x_2 \end{vmatrix} = 1x_1 + 1x_2$

$\Rightarrow n = (1,1)^T$

$\Rightarrow V = \frac{1}{1}\|n\| = \sqrt{2}$




$d_{2,1} = (-1, 1, 0)^T$

$d_{3,1} = (-1, 0, 1)^T$

$\Rightarrow D = \begin{vmatrix} -1 & -1 & x_1 \\ 1 & 0 & x_2 \\ 0 & 1 & x_3 \end{vmatrix} = \begin{matrix} 1x_1 \\ +1x_2 \\ +1x_3 \end{matrix}$

$\Rightarrow n = (1,1,1)^T$

$\Rightarrow V = \frac{1}{2}\|n\| = \frac{\sqrt{3}}{2}$

• **General:** $V = \frac{1}{d!}\|n\|$.

• **Note.** Cramer's rule allows one to compute the solution of a linear system of equations via determinants. A system with $n$ unknowns and $n$ given equations — $Ax = y$ — has the solution vector $x$ with components $x_i = \det A_i / \det A$. The column vectors of $A_i$ are identical to those of $A$, except its $i$-th column which is defined as $y$. Generally, this solution method based on Cramer's rule is computationally inefficient for large values of $n$, unless the matrix $A$ has a "favorable structure" or a solution does not need to be computed for a "real-time application." The specific example presented on the previous page involved the normal equations $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. This system leads to the determinants $\det A = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3$, $\det A_1 = \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 1$ and $\det A_2 = \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = 1$. Thus $u_2 = \det A_1 / \det A = \frac{1}{3}$ and $u_3 = \det A_2 / \det A = \frac{1}{3}$ (and $u_1 = 1 - \frac{2}{3} = \frac{1}{3}$).

• **Note.** Determinants can also be used to compute orientation-dependent, signed hyper-volumes of simplices in arbitrary dimension, see figure (left)....