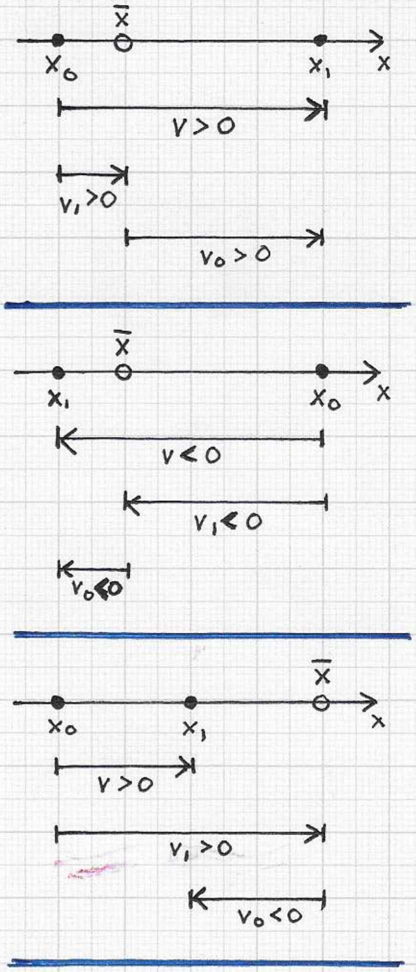
■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions: ... It is important to always realize that INDEXING–ORDER–DIRECTION–ORIENTATION impacts the values and signs of barycentric coordinates. We discuss this aspect in detail for a few lower-dimensional cases. The figures (left) show three examples for the one-dimensional ("x-axis") case. A reference line segment is defined by vertices x_0 and x_1, and the point $\bar{x}$ must be expressed relative to x_0 and x_1. The three examples show the cases $x_0 < \bar{x} < x_1$; $x_1 < \bar{x} < x_0$; and $x_0 < x_1 < \bar{x}$. The barycentric coordinates of $\bar{x}$ are calculated formally as follows:

Barycentric coordinates and "volumes" on the x-axis:



Illustrations of three examples of arrangements of simplex vertices x_0 and x_1 and a specific point $\bar{x}$ on the x-axis. Ratios of the indicated "volumes" V, V_0 and V_1 define the barycentric coordinates of $\bar{x}$ relative to the reference simplex/line segment given by x_0 and x_1.

$$\bar{x} = u_0 x_0 + u_1 x_1 \qquad (\text{with } u_0 = 1-u_1)$$
$$= (1-u_1)\,x_0 + u_1 x_1$$
$$= x_0 + u_1 x_1 - u_1 x_0$$
$$\Rightarrow (x_1 - x_0)\,u_1 = \bar{x} - x_0$$
$$u_1 = \frac{\det(\bar{x}-x_0)}{\det(x_1-x_0)} = \frac{V_1}{V} = \frac{\bar{x}-x_0}{x_1-x_0}$$

Or: $\bar{x} = u_0 x_0 + (1-u_0)\,x_1$
$$\Rightarrow (x_0 - x_1)\,u_0 = \bar{x} - x_1$$
$$u_0 = \frac{\det(\bar{x}-x_1)}{\det(x_0-x_1)} = \frac{V_0}{V} \quad \frac{x_1-\bar{x}}{x_1-x_0}.$$
$$\frac{\det(\bar{x}-x_1)}{\det(x_0-x_1)} = \frac{\det(x_1-\bar{x})}{\det(x_1-x_0)}$$

...

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions: ... The relationship between barycentric coordinates, signed simplex volumes and determinants becomes evident when formulating the calculation of barycentric coordinates via a linear equation system. We consider the 1D case:

$$\Rightarrow \quad \bar{x} = u_0 x_0 + u_1 x_1$$
$$1 = u_0 + u_1$$

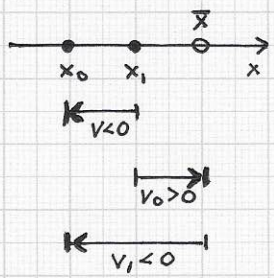
When using Cramer'rule to solve this linear system, one obtains the determinants

$$D = \begin{vmatrix} x_0 & x_1 \\ 1 & 1 \end{vmatrix} = x_0 - x_1 \; ; \; D_0 = \begin{vmatrix} \bar{x} & x_1 \\ 1 & 1 \end{vmatrix} = \bar{x} - x_1 \; ;$$

$$D_1 = \begin{vmatrix} x_0 & \bar{x} \\ 1 & 1 \end{vmatrix} = x_0 - \bar{x} \, .$$

Thus, $u_0 = D_0 / D = \frac{\bar{x} - x_1}{x_0 - x_1}$; $u_1 = D_1 / D = \frac{x_0 - \bar{x}}{x_0 - x_1}$.

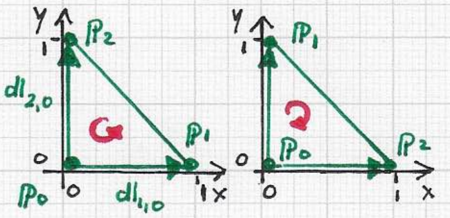
For example, using the values $x_0 = 0$, $x_1 = 1$ and $\bar{x} = 2$ yields the barycentric coordinate values $u_0 = \frac{1}{-1} = -1$ and $u_1 = \frac{-2}{-1} = 2$.



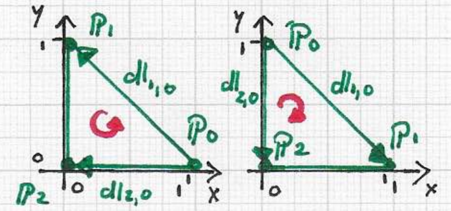
The signed simplex volumes / sub-simplex volumes are defined by the determinants D, D_0 and D_1, i.e., $V = D$, $V_0 = D_0$ and $V_1 = D_1$. The "more common formulas" $u_0 = \frac{x_1 - \bar{x}}{x_1 - x_0}$ and $u_1 = \frac{\bar{x} - x_0}{x_1 - x_0}$ are obtained by multiplying D, D_0 and D_1 by (-1).

...

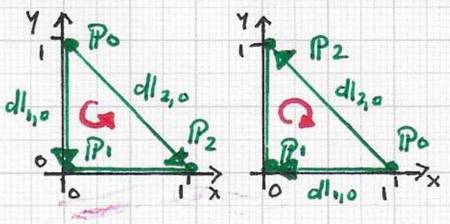
Barycentric coordinates and "volumes" in the xy-plane:



$V = \frac{1}{2} \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = \frac{1}{2}$ $\quad V = \frac{1}{2} \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -\frac{1}{2}$



$V = \frac{1}{2} \begin{vmatrix} -1 & -1 \\ 1 & 0 \end{vmatrix} = \frac{1}{2}$ $\quad V = \frac{1}{2} \begin{vmatrix} 1 & 0 \\ -1 & -1 \end{vmatrix} = -\frac{1}{2}$



$V = \frac{1}{2} \begin{vmatrix} 0 & 1 \\ -1 & -1 \end{vmatrix} = \frac{1}{2}$ $\quad V = \frac{1}{2} \begin{vmatrix} -1 & -1 \\ 0 & 1 \end{vmatrix} = -\frac{1}{2}$

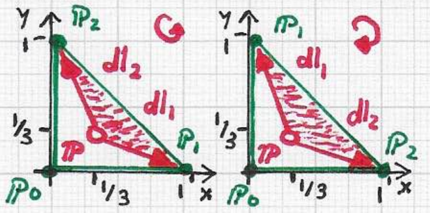
The six possible ways to index the vertices of a triangle with volume 1/2 or −1/2, depending on counter-clockwise or clockwise orientation of the three vertices.

Any point P in the xy-plane can be used to split the "reference triangle" into three sub-triangles; their signed volumes define P's coordinates.

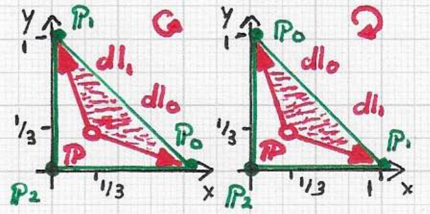
■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions: … The figures on the previous and this page (left) show the influence of indexing / orientation on the values of determinants – their signs – in the case of signed "volume" computations for triangles in the xy-plane:

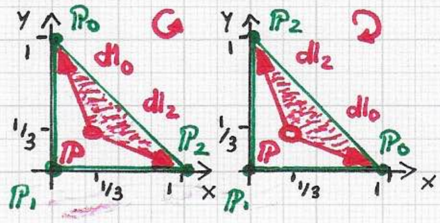
Barycentric coordinates and "volumes" in the xy-plane – sub-simplex "volumes":



$$V_0 = \frac{1}{2}\begin{vmatrix} 2/3 & -1/3 \\ -1/3 & 2/3 \end{vmatrix} = \frac{1}{6} \qquad V_0 = \frac{1}{2}\begin{vmatrix} -1/3 & 2/3 \\ 2/3 & -1/3 \end{vmatrix} = -\frac{1}{6}$$



$$V_2 = \frac{1}{2}\begin{vmatrix} 2/3 & -1/3 \\ -1/3 & 2/3 \end{vmatrix} = \frac{1}{6} \qquad V_2 = \frac{1}{2}\begin{vmatrix} -1/3 & 2/3 \\ 2/3 & -1/3 \end{vmatrix} = -\frac{1}{6}$$



$$V_1 = \frac{1}{2}\begin{vmatrix} 2/3 & -1/3 \\ -1/3 & 2/3 \end{vmatrix} = \frac{1}{6} \qquad V_1 = \frac{1}{2}\begin{vmatrix} -1/3 & 2/3 \\ 2/3 & -1/3 \end{vmatrix} = -\frac{1}{6}$$

The six possible way to index the vertices of the "upper-right sub-triangle" with "volume" 1/6 or −1/6, depending on orientation.

The other two sub-triangles also have "volumes" of values 1/6 or −1/6.

The value of barycentric coordinate u_i is defined as $u_i = v_i / v$, $i = 0, 1, 2$.

A point $\mathbb{P}$ is represented via barycentric coordinates relative to the three vertices of a reference simplex (triangle). The specific point $\mathbb{P} = (1/3, 1/3)^T$ is represented relative to the points with coordinate tuples $(0,0)^T$, $(1,0)^T$ and $(0,1)^T$. IT IS IMPORTANT TO REALIZE THAT THE BARYCENTRIC COORDINATE TUPLE IS ALWAYS $(1/3, 1/3, 1/3)^T$, regardless of the chosen indexing / orientation of vertices / triangles. **!**

A coordinate value is defined either as $u_i = \frac{1}{6} : \frac{1}{2} = \frac{1}{3}$ (counter-clockwise case) or $u_i = (-\frac{1}{6}) : (-\frac{1}{2}) = \frac{1}{3}$ (clockwise case). In other words, the value and sign of a point's barycentric coordinate relative to a specific reference point is independent of all reference points' indexing and simplex orientation. …

OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions:

Barycentric coordinates in the xy-plane – the determinants involved:

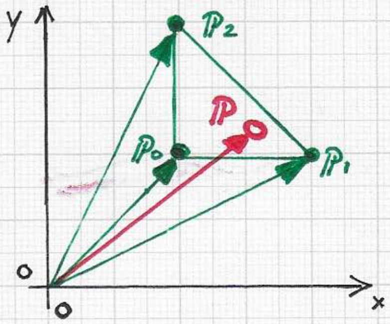
$$D = \begin{vmatrix} 1 & 1 & 1 \\ x_0 & x_1 & x_2 \\ y_0 & y_1 & y_2 \end{vmatrix} = \begin{matrix} x_1 y_2 - x_2 y_1 \\ - x_0 y_2 + x_2 y_0 \\ + x_0 y_1 - x_1 y_0 \end{matrix}$$

$$D_0 = \begin{vmatrix} 1 & 1 & 1 \\ \bar{x} & x_1 & x_2 \\ \bar{y} & y_1 & y_2 \end{vmatrix} = \begin{matrix} x_1 y_2 - x_2 y_1 \\ - \bar{x} y_2 + x_2 \bar{y} \\ + \bar{x} y_1 - x_1 \bar{y} \end{matrix}$$

$$D_1 = \begin{vmatrix} 1 & 1 & 1 \\ x_0 & \bar{x} & x_2 \\ y_0 & \bar{y} & y_2 \end{vmatrix} = \begin{matrix} \bar{x} y_2 - x_2 \bar{y} \\ - x_0 y_2 + x_2 y_0 \\ + x_0 \bar{y} - \bar{x} y_0 \end{matrix}$$

$$D_2 = \begin{vmatrix} 1 & 1 & 1 \\ x_0 & x_1 & \bar{x} \\ y_0 & y_1 & \bar{y} \end{vmatrix} = \begin{matrix} x_1 \bar{y} - \bar{x} y_1 \\ - x_0 \bar{y} + \bar{x} y_0 \\ + x_0 y_1 - x_1 y_0 \end{matrix}$$

The needed four determinants when using Cramer's rule to compute the values of u_0, u_1, u_2.



Data involved in the computation of the determinants D, D_0, D_1 and D_2: Simplex / triangle vertices P_0, P_1, P_2 and the point P. The data can be viewed as point data or vector data – by considering the points' positional vectors. **The terms in the determinant expressions are related to cross products of positional vectors.**

• **Note.** The "more common formulas" for the barycentric coordinates of $\bar{x}$ on the x-axis, relative to x_0 and x_1, are obtained by merely altering the order of the two linear equations:

$$\boxed{\begin{matrix} u_0 & + & u_1 & = 1 \\ x_0 u_0 & + & x_1 u_1 & = \bar{x} \end{matrix}} \Rightarrow \quad D = \begin{vmatrix} 1 & 1 \\ x_0 & x_1 \end{vmatrix} = x_1 - x_0 , \quad D_0 = \begin{vmatrix} 1 & 1 \\ \bar{x} & x_1 \end{vmatrix} = x_1 - \bar{x} , \quad D_1 = \begin{vmatrix} 1 & 1 \\ x_0 & \bar{x} \end{vmatrix} = \bar{x} - x_0 .$$

"Order of equations matters."

$$\Rightarrow u_0 = \frac{x_1 - \bar{x}}{x_1 - x_0} , \quad u_1 = \frac{\bar{x} - x_0}{x_1 - x_0} .$$

Considering this **Note**, a purely algebraic approach to calculating the three unknown values of barycentric coordinates u_0, u_1 and u_2 of a point $P = (\bar{x}, \bar{y})^T$ with respect to a "reference triangle" with vertices $P_i = (x_i, y_i)^T$, $i = 0,1,2$, leads to the linear equation system

$$\boxed{\begin{matrix} u_0 + & u_1 + & u_2 & = & 1 \\ x_0 u_0 + & x_1 u_1 + & x_2 u_2 & = & \bar{x} \\ y_0 u_0 + & y_1 u_1 + & y_2 u_2 & = & \bar{y} \end{matrix}} \cdot \quad \begin{bmatrix} 1 & 1 & 1 \\ x_0 & x_1 & x_2 \\ y_0 & y_1 & y_2 \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 \\ \bar{x} \\ \bar{y} \end{bmatrix} .$$

The u_i-values can be computed via Cramer's rule, see left-hand side, and they are $u_i = D_i / D$, $i = 0,1,2$. Interestingly, the terms defining the determinants also arise in cross products. ...

OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions:... In the case of these **3-by-3** determinants $\mathcal{D}$, $\mathcal{D}_0$, $\mathcal{D}_1$ and $\mathcal{D}_2$ it is possible to interpret the determinant terms via cross products of vectors in 3D space. We consider $\mathcal{D}$ in detail:

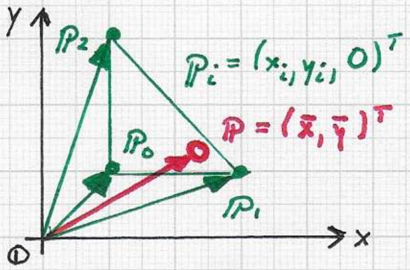
$$\mathcal{D} = (+x_1y_2 - x_2y_1) + (-x_0y_2 + x_2y_0) + (x_0y_1 - x_1y_0) = W_{12} + W_{20} + W_{01}$$

$$\mathcal{D}_0 = (+x_1y_2 - x_2y_1) + (-\bar{x}y_2 + x_2\bar{y}) + (\bar{x}y_1 - x_1\bar{y}) = W_{12} + W_{2\bullet} + W_{\bullet 1}$$

$$\mathcal{D}_1 = (+\bar{x}y_2 - x_2\bar{y}) + (-x_0y_2 + x_2y_0) + (x_0\bar{y} - \bar{x}y_0) = W_{\bullet 2} + W_{20} + W_{0\bullet}$$

$$\mathcal{D}_2 = (+x_1\bar{y} - \bar{x}y_1) + (-x_0\bar{y} + \bar{x}y_0) + (x_0y_1 - x_1y_0) = W_{1\bullet} + W_{\bullet 0} + W_{01}$$

Positional vectors and cross products defining determinants:



• Three cross products:

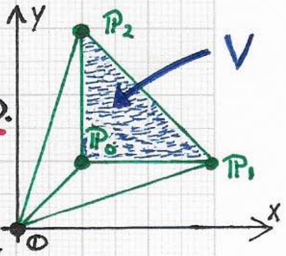
$\mathbb{p}_0 \times \mathbb{p}_1$, $\mathbb{p}_1 \times \mathbb{p}_2$, $\mathbb{p}_2 \times \mathbb{p}_0$

$$\mathbb{p}_0 \times \mathbb{p}_1: \begin{vmatrix} x_0 & x_1 & x \\ y_0 & y_1 & y \\ 0 & 0 & z \end{vmatrix} = z(x_0y_1 - x_1y_0) = z\,w_{01}$$

$$\mathbb{p}_1 \times \mathbb{p}_2: \begin{vmatrix} x_1 & x_2 & x \\ y_1 & y_2 & y \\ 0 & 0 & z \end{vmatrix} = z(x_1y_2 - x_2y_1) = z\,w_{12}$$

$$\mathbb{p}_2 \times \mathbb{p}_0: \begin{vmatrix} x_2 & x_0 & x \\ y_2 & y_0 & y \\ 0 & 0 & z \end{vmatrix} = z(x_2y_0 - x_0y_2) = z\,w_{20}$$

We can consider the geometrical interpretation of $\mathcal{D}$. The "reference simplex" with vertices $\mathbb{p}_0, \mathbb{p}_1, \mathbb{p}_2$ has the "volume" V. And: $V = \mathrm{vol}(\mathrm{simp}(\mathbb{O}, \mathbb{p}_1, \mathbb{p}_2)) - \mathrm{vol}(\mathrm{simp}(\mathbb{O}, \mathbb{p}_0, \mathbb{p}_2)) - \mathrm{vol}(\mathrm{simp}(\mathbb{O}, \mathbb{p}_1, \mathbb{p}_0))$.* ...



• Three addt'l cross products:

$\mathbb{p} \times \mathbb{p}_0$, $\mathbb{p} \times \mathbb{p}_1$, $\mathbb{p} \times \mathbb{p}_2$

$$\mathbb{p} \times \mathbb{p}_0: \begin{vmatrix} \bar{x} & x_0 & x \\ \bar{y} & y_0 & y \\ 0 & 0 & z \end{vmatrix} = z(\bar{x}y_0 - x_0\bar{y}) = z\,W_{\bullet 0}$$

$$\Rightarrow \mathbb{p}_0 \times \mathbb{p} = z\,W_{0\bullet} = -z\,W_{\bullet 0}$$

$$\mathbb{p} \times \mathbb{p}_1: \begin{vmatrix} \bar{x} & x_1 & x \\ \bar{y} & y_1 & y \\ 0 & 0 & z \end{vmatrix} = z(\bar{x}y_1 - x_1\bar{y}) = z\,W_{\bullet 1}$$

$$\Rightarrow \mathbb{p}_1 \times \mathbb{p} = z\,W_{1\bullet} = -z\,W_{\bullet 1}$$

$$\mathbb{p} \times \mathbb{p}_2: \begin{vmatrix} \bar{x} & x_2 & x \\ \bar{y} & y_2 & y \\ 0 & 0 & z \end{vmatrix} = z(\bar{x}y_2 - x_2\bar{y}) = z\,W_{\bullet 2}$$

$$\Rightarrow \mathbb{p}_2 \times \mathbb{p} = z\,W_{2\bullet} = -z\,W_{\bullet 2}$$

* $\mathrm{simp}(\mathbb{a}, \mathbb{b}, \mathbb{c})$ = simplex with **ordered** vertices $\mathbb{a}, \mathbb{b}, \mathbb{c}$.