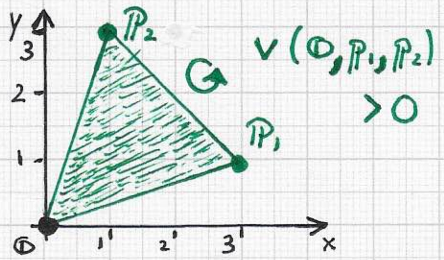
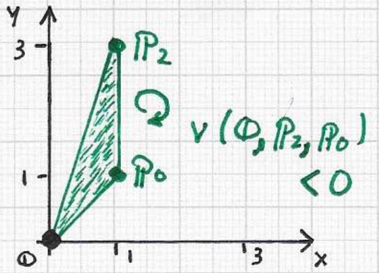
■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions: ... Points/positional vectors, determinants and signed volumes of simplices:

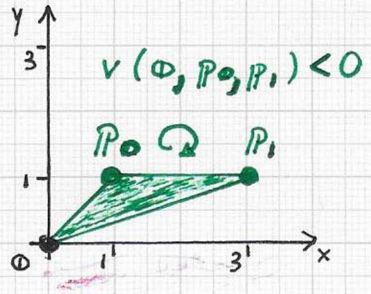
$V = 4$



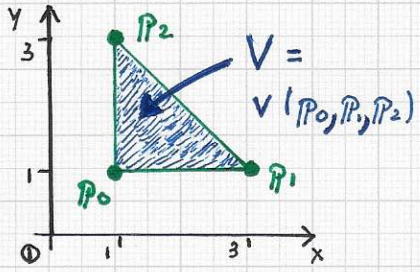
$v = -1$



$v = -1$



$V = 4 - 1 - 1 = 2$



Example of the relationship between the volume V of the simplex with corner points $\mathbb{P}_0, \mathbb{P}_1, \mathbb{P}_2$ and the associated simplices with the origin and two positional vectors: $\mathbb{P}_1, \mathbb{P}_2$ and $\mathbb{P}_2, \mathbb{P}_0$ and $\mathbb{P}_0, \mathbb{P}_1$.

It is interesting and important to see these intricate and not immediately obvious relationships between corner points and their positional vectors of a simplex; signed simplex volume; ordering/indexing/orientation; determinants; and cross products of vectors (in the 3D case). We consider the determinant $\mathcal{D}$ as defined on the previous page for points/positional vectors $\mathbb{P}_i = (x_i, y_i)^T$, $i = 0,1,2$:

$$\mathcal{D} = w_{12} + w_{20} + w_{01} \ .$$

For illustrative purposes, we use the specific points $\mathbb{P}_0 = (1,1)^T$, $\mathbb{P}_1 = (3,1)^T$ and $\mathbb{P}_2 = (1,3)^T$, see figures (left). The resulting w_{ij}-values for these three points (simplex corner points) are $w_{12} = 8$, $w_{20} = -2$ and $w_{01} = -2$. These values are the z-coordinate values of the three corresponding cross products $\mathbb{P}_1 \times \mathbb{P}_2$, $\mathbb{P}_2 \times \mathbb{P}_0$ and $\mathbb{P}_0 \times \mathbb{P}_1$. Therefore, these values represent twice the values of the signed volumes (= areas) of the simplices $\mathbb{O}\mathbb{P}_1\mathbb{P}_2$, $\mathbb{O}\mathbb{P}_2\mathbb{P}_0$ and $\mathbb{O}\mathbb{P}_0\mathbb{P}_1$, called $v(\mathbb{O}, \mathbb{P}_1, \mathbb{P}_2)$, $v(\mathbb{O}, \mathbb{P}_2, \mathbb{P}_0)$ and $v(\mathbb{O}, \mathbb{P}_0, \mathbb{P}_1)$. The value of $V = v(\mathbb{P}_0, \mathbb{P}_1, \mathbb{P}_2)$ is the sum of the three v-values involving $\mathbb{O}$ as an argument. ...

Stratovan

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions: ... Thus, the value of determinant $\mathbb{D}$ is

$$\mathbb{D} = w_{12} + w_{20} + w_{01} = 8 - 2 - 2 = 4.$$

Consequently, the signed volume of simplex $\mathbb{P}_0\mathbb{P}_1\mathbb{P}_2$ is

$$V = \tfrac{1}{2}\,\mathbb{D} = \tfrac{1}{2}\cdot 4 = 2.$$

The point to be represented relative to the reference simplex $\mathbb{P}_0\mathbb{P}_1\mathbb{P}_2$ is $\mathbb{P} = (3,3)^T$. We compute the values of $\mathbb{D}_0$, $\mathbb{D}_1$, and $\mathbb{D}_2$. The value of $\mathbb{D}_0$ is

$$\mathbb{D}_0 = w_{12} + w_{2\bullet} + w_{\bullet 1} = 8 - 6 - 6 = -4.$$

Thus, the signed volume of simplex $\mathbb{P}\mathbb{P}_1\mathbb{P}_2$ is

$$V_0 = \tfrac{1}{2}\,\mathbb{D}_0 = -2.$$

The figures (left) illustrate the geometrical meaning of some of the terms involved in the computations. The value of $\mathbb{D}_1$ is

$$\mathbb{D}_1 = w_{\bullet 2} + w_{20} + w_{0\bullet} = 6 - 2 + 0 = 4,$$

and the signed volume of simplex $\mathbb{P}\mathbb{P}_2\mathbb{P}_0$ is

$$V_1 = \tfrac{1}{2}\,\mathbb{D}_1 = 2.$$

The value of $\mathbb{D}_2$ is

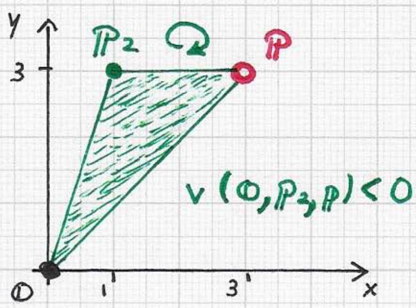
$$\mathbb{D}_2 = w_{1\bullet} + w_{\bullet 0} + w_{01} = 6 + 0 - 2 = 4,$$

and the signed volume of simplex $\mathbb{P}\mathbb{P}_0\mathbb{P}_1$ is

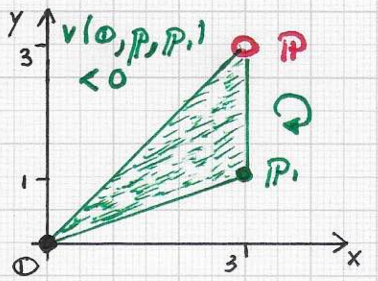
$$V_2 = \tfrac{1}{2}\,\mathbb{D}_2 = 2,$$

see figures (next page). ...

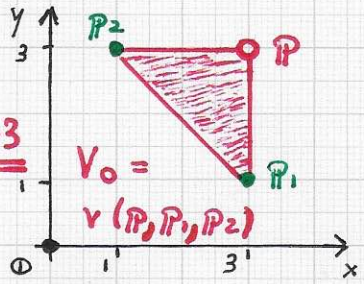
$V = -3$



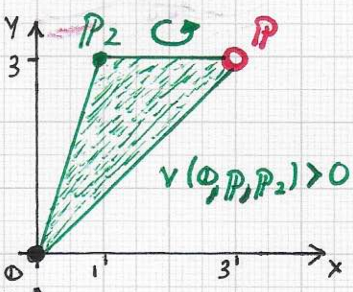
$V = -3$



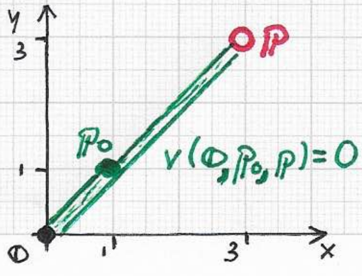
$V_0 = 4 - 3 - 3 = -2$

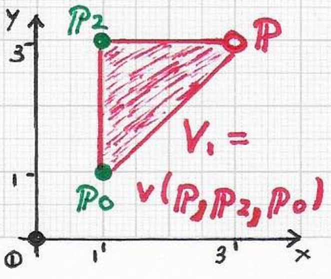


$V = 3$



$V = 0$





$V_1 = 3 - 1 + 0 = 2$

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions: ... In summary, these detailed computations have produced the determinant values

$$D = 4,\ D_0 = -4,\ D_1 = 4,\ \text{and}\ D_2 = 4$$

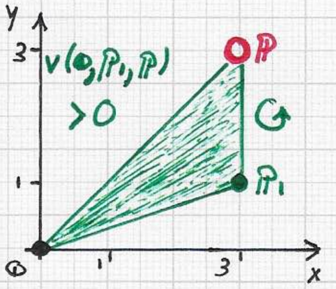
and the corresponding signed simplex volume values

$$V = 2,\ V_0 = -2,\ V_1 = 2,\ \text{and}\ V_2 = 2.$$

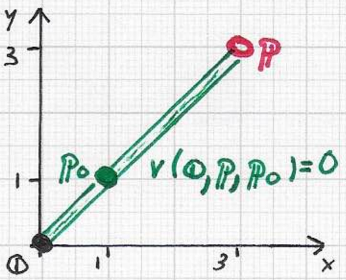
The resulting barycentric coordinate tuple (u_0, u_1, u_2) of $\mathbb{P}$ is defined by signed determinant or signed simplex volume ratios:

$$(u_0, u_1, u_2) = (D_0/D, D_1/D, D_2/D) = (-1, 1, 1) = (V_0/V, V_1/V, V_2/V).$$

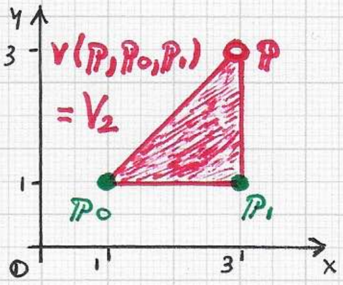
V = 3



V = 0



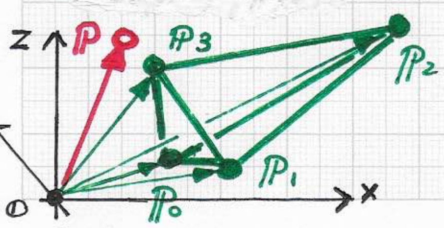
$V_2 = 3 + 0 - 1 = 2$



The relationships between the volume V_i of the simplex with corner points $\mathbb{P}, \mathbb{P}_{i+1}, \mathbb{P}_{i+2}$, $i = 0, 1, 2$ (index values modulo 3) and the associated simplices with the origin and two positional vectors $\mathbb{P}_{i+1}, \mathbb{P}_{i+2}$.

The most important insight obtained from this detailed discussion and example concerns the relationship between ratios of determinants (signed simplex volumes) and barycentric coordinate values.

Vector calculus / multivariate calculus provides another **relevant** approach to simplex volume computation: **INTEGRATION VIA THE "CHANGE-OF-VARIABLES THEOREM."** ...



Data involved in the computation of the barycentric coord. of $\mathbb{P} = (\bar{x}, \bar{y}, \bar{z})^T$.

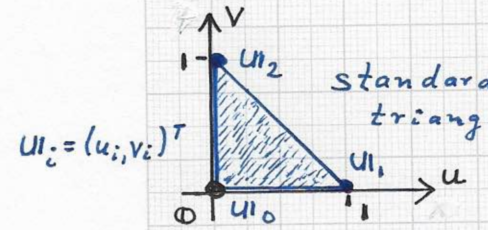
■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

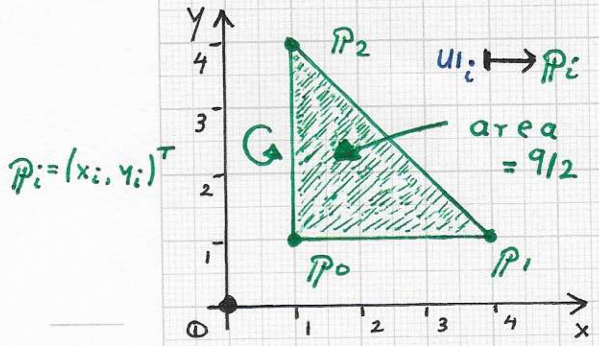
• Laplacian eigenfunctions:... The change-of-variables theorem makes it possible to simplify integration over a "complicated domain" to integration over a "simple domain." This is done by using a differentiable mapping that maps the "simple" to the "complicated domain." (For example, see "Vector Calculus" by J. Marsden and A. Tromba.) We are concerned with the computation of signed simplex volumes in any dimension; and volume computation can be done via integration. Further, any simplex can be obtained via a simple linear mapping of some simple "standard simplex." Thus, we can employ this theorem for our purpose.

The mapping $M: \mathbb{R}^k \to \mathbb{R}^k$ is a C^1-continuous mapping used to map a region $R \subset \mathbb{R}^k$ to a region $\bar{R} \subset \mathbb{R}^k$. The Jacobian (determinant) of the mapping is defined by the mapping's partial derivatives:

$$J = \begin{vmatrix} \partial x_1/\partial u_1 & \dots & \partial x_1/\partial u_k \\ \vdots & & \vdots \\ \partial x_k/\partial u_1 & \dots & \partial x_k/\partial u_k \end{vmatrix}.$$

...





Example of a simple linear mapping, i.e., a concatenation of scaling and translation:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

The Jacobian J is

$$J = \begin{vmatrix} 3 & 0 \\ 0 & 3 \end{vmatrix} = 9.$$

One can compute the area of triangle $\mathbb{p}_0\mathbb{p}_1\mathbb{p}_2$ directly; it is

$$\int_{x=1}^{4} \int_{y=1}^{5-x} 1 \, dy \, dx = \frac{9}{2}.$$

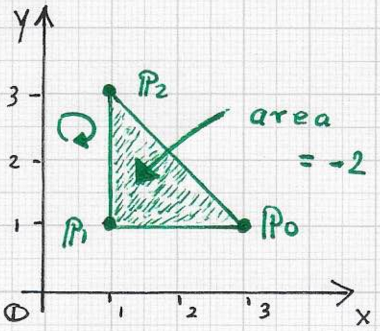
Using the change-of-variables theorem, one obtains

$$\int_{u=0}^{1} \int_{v=0}^{1-u} 1 \cdot J \, dv \, du = \frac{9}{2}.$$

OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions: ... The coordinate functions of a point $\mathbb{x} = (x_1, \ldots, x_k)^T$ are the result of mapping the point $\mathbb{u} = (u_1, \ldots, u_k)^T$ to $\mathbb{x}$, i.e., $x_1 = x_1(u_1, \ldots, u_k)^T, \ldots, x_k = x_k(u_1, \ldots, u_k)^T$. Thus, the k^2 first partial derivatives define the Jacobian J. Integration over the region $\overline{R}$ in $\mathbb{x}$-space can be simplified to integration over the region R in $\mathbb{u}$-space:

$$\int_{\overline{R}} f(\mathbb{x})\, d\mathbb{x} = \int_R f(\mathbb{x}(\mathbb{u}))\, J\, d\mathbb{u}.$$



Another example:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 & -2 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$\Rightarrow J = \begin{vmatrix} -2 & -2 \\ 0 & 2 \end{vmatrix} = -4$$

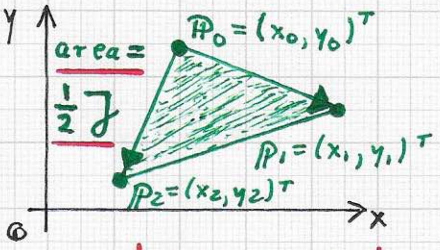
$$\Rightarrow \text{area}(P_0, P_1, P_2) = J \cdot \int_{u=0}^{1} \int_{v=0}^{1-u} 1\, dv\, du = -4 \cdot \frac{1}{2} = -2.$$

The change-of-variables theorem leads to the CORRECTLY SIGNED simplex volumes, depending on orientation.

We only need to consider Linear mappings that can be written as

$$\begin{bmatrix} x_1 = x_1(\mathbb{u}) \\ \vdots \\ x_k = x_k(\mathbb{u}) \end{bmatrix} = \begin{bmatrix} a_{1,1} & \cdots & a_{1,k} \\ \vdots & & \vdots \\ a_{k,1} & \cdots & a_{k,k} \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ u_k \end{bmatrix} + \begin{bmatrix} b_1 \\ \vdots \\ b_k \end{bmatrix}$$

$$\Leftrightarrow \mathbb{x}(\mathbb{u}) = A\, \mathbb{u} + \mathbb{b}.$$



Thus, the Jacobian J is

$$J = \det(A) = \begin{vmatrix} a_{1,1} & \cdots & a_{1,k} \\ \vdots & & \vdots \\ a_{k,1} & \cdots & a_{k,k} \end{vmatrix}.$$

$$J = \begin{vmatrix} x_1 - x_0 & x_2 - x_0 \\ y_1 - y_0 & y_2 - y_0 \end{vmatrix}$$

Further, for simple volume computations, the function f is the constant function $f = 1$. ...

General linear mapping.