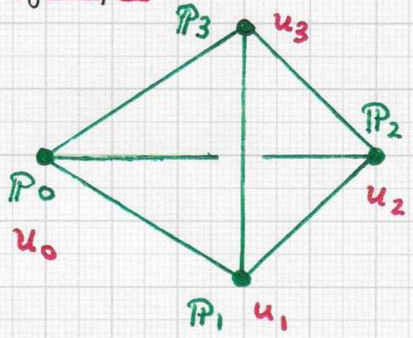
OBJECT AND MATERIAL EIGENFUNCTIONS — Cont'd.

• Laplacian eigenfunctions: ... Despite the fact that barycentric coordinate values can be negative, we can still interpret the value of a barycentric coordinate u_i of point p_i as a "probability measure" indicating whether the point $p = \sum_{j=0}^{k-1} u_j p_j$ is likely to represent the "feature data" reflected by p_i (or not). The figure (left) is an illustration of a 3-simplex. This simplex defines lower-dimensional sub-simplices of dimensions 0, 1 and 2, i.e., the 4 individual vertices, the 6 edges and the 4 triangle faces. Thus, the total number of simplices — 0-simplices, 1-simplices, ..., (k-1)-simplices — that can be defined by k points $p_0, \ldots, p_{k-1}$ is 2^k-1, given by the sum $\binom{k}{1}+\binom{k}{2}+\ldots+\binom{k}{k}$. In the context of classification, **why is it of interest to possibly consider such lower-dimensional sub-simplices of a (k-1)-simplex?** For example, the given vertices p_i can represent "feature data" of DIFFERENT MATERIAL SAMPLES belonging to the SAME MATERIAL CLASS. ...

Abstract representation of a 3-simplex with 4 vertices that lie on a hyper-sphere in 3-dimensional space, shown as a complete graph:



Writing the index tuple of this simplex as (0,1,2,3), one can use this notation to also list the index tuples of all "sub-simplices" of all possible lower dimensions D, i.e., D=0, D=1, D=2:

D=0: (0), (1), (2), (3)

D=1: (0,1), (0,2), (0,3), (1,2), (1,3), (2,3)

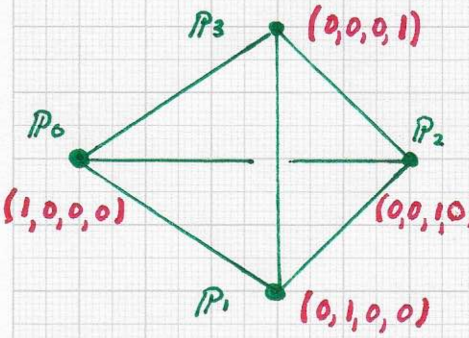
D=2: (0,1,2), (0,1,3), (0,2,3), (1,2,3)

Counting the "empty simplex" and the shown 3-simplex itself, the total number of sub-simplices is 16, i.e., $1+4+6+4+1=2^4$.

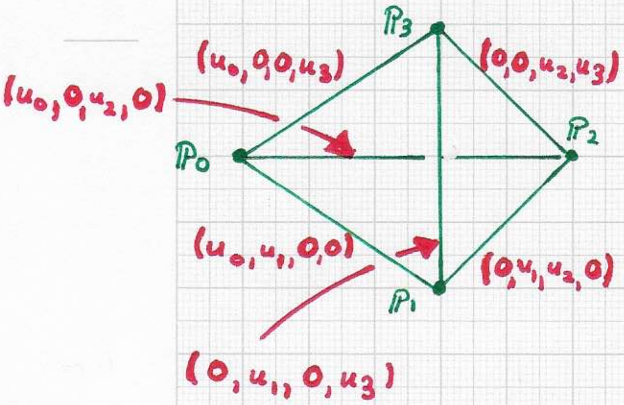
OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions: ... In this discussion, we can view term "point p_j" as a synonym for the term "class (or sample) j" – since p_j represents derived feature data that reflect a characteristic of a class-j sample. Again, one might have only one or multiple samples that belong to a material class j. Given the barycentric coordinate tuple $(u_0, u_1, \ldots, u_{k-1})$ of an "unclassified point p," and interpreting the u_i-values as quasi-probability values, one can use the tuple for the purpose of answering the following kinds of classification questions:

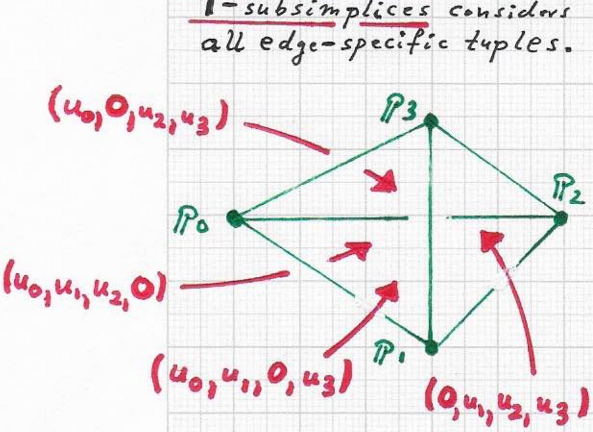
- • Does p belong to the class of p_0?
- • " p " " " " " p_1?
- • " p " " " " " p_{k-1}?
- •• Does p belong to the class (p_0 or p_1)?
- •• " p " " " " (p_0 or p_{k-1})?
- •• " p " " " " (p_{k-2} or p_{k-1})?
- ••• Does p belong to the class (p_0 or p_1 or p_2)?
- ••• " p " " " " (p_0 or p_{k-2} or p_{k-1})?
- ••• " p " " " " (p_{k-3} or p_{k-2} or p_{k-1})?



Classification done via 0-subsimplices considers only the vertex-specific coordinate tuples of p_j.



Classification done via 1-subsimplices considers all edge-specific tuples.



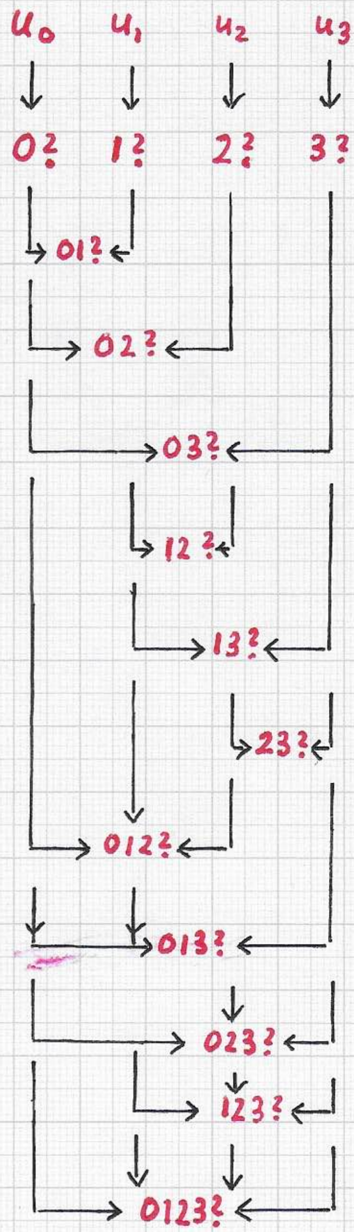
Classification done via 2-subsimplices considers all face-specific tuples.

...

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd

• Laplacian eigenfunctions: ... The questions asked on the previous page (bottom) are merely exemplary. The questions marked with •, •• or ∴ concern classification possibilities that would determine whether a new, unclassified point $\mathbb{P}$ with barycentric coordinate tuple $(u_0, u_1, \ldots, u_{k-1})$ can be viewed as being of a specific "vertex class" $\mathbb{P}_j$, "edge class" $\overline{\mathbb{P}_j \mathbb{P}_k}$ or "face class" $\Delta\, \mathbb{P}_j \mathbb{P}_k \mathbb{P}_l$, respectively. One can also interpret a barycentric coordinate u_i as a weight, i.e., as a weight that is a measure of similarity between the unclassified point $\mathbb{P}$ and the classified point having the index i. When each vertex $\mathbb{P}_j$ of a simplex represents a DIFFERENT material class, considering 1-subsimplices, 2-subsimplices etc. does not seem to be meaningful: One should determine only whether the unclassified point $\mathbb{P}$ does or does not belong to one of the different classes described by one of the simplex vertices.

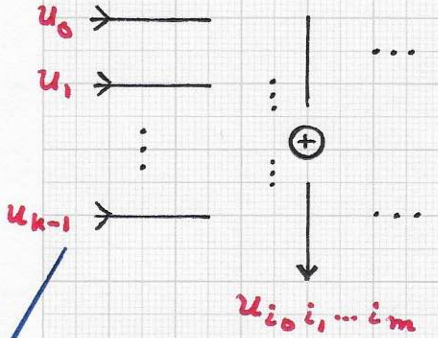
...



Four barycentric coordinates allow one to consider $2^4 - 1 = 15$ classification problems: (i) Is it class 0? ("0?") ... (v) Is it class 0 OR class 1? ("01?") ... (xv) Is it class 0 OR class 1 OR class 2 OR class 3? ("0123?").

■ OBJECT AND MATERIAL EIGENFUNCTIONS — Cont'd.

• Laplacian eigenfunctions: ... Of course, the vertices of the simplex can also be "sample representatives" of the SAME material class. In this scenario, one can view the simplex vertices as a finite "training data set." It is therefore meaningful to also view "space between vertices and in close vicinity of the convex simplex" as space that represents the material class exemplified by the finite simplex vertex set. Thus, the space between two vertices, a 1-subsimplex, the space between three vertices, a 2-subsimplex, etc. should be viewed as space representing the same material class — at least with a certain significant level of probability. The figure (bottom) shows how a point's 4 coordinates u_0, u_1, u_2, u_3 can be added in $2^4-1=15$ combinatorially possible ways.



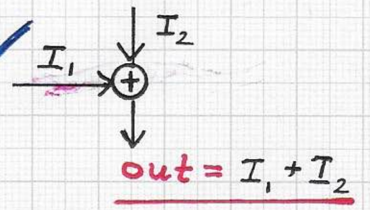
Computation of an additive coordinate value:

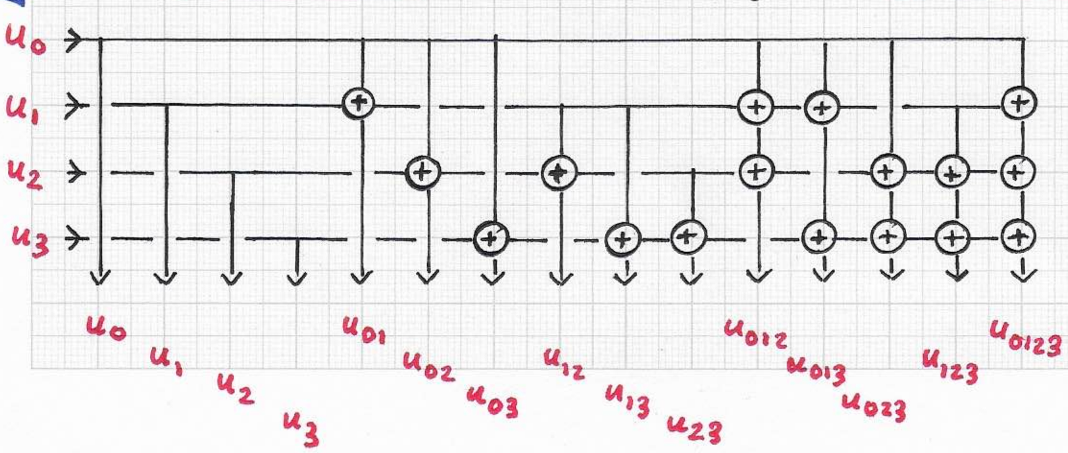
$$u_{i_0 i_1 \cdots i_m} = \sum_{\ell=0}^{m} u_{i_\ell},$$

where

$i_0, i_1, \ldots, i_m \in \{0,1,\ldots,k-1\}$, $0 \le m \le k-1$ and i_ℓ-values are different from each other.

meaning of network

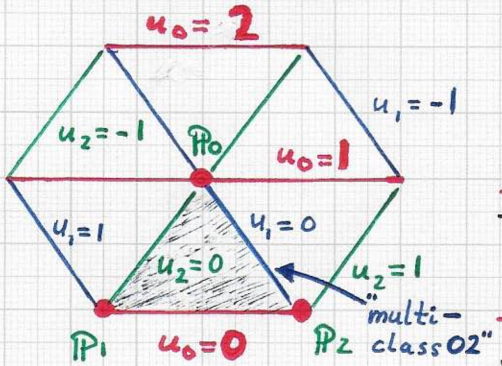




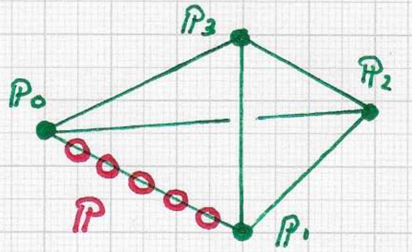
...

■ OBJECT AND MATERIAL EIGENFUNCTIONS — Cont'd.

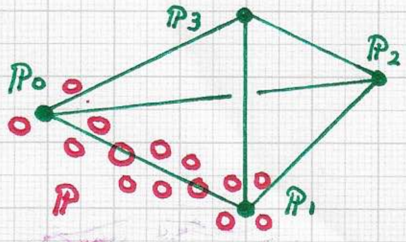
• Laplacian eigenfunctions: … If one were to follow postulates of probability theory strictly, one could not refer to the barycentric coordinate u_i of a point p as probability (that p's probability value for being of class i is the value of u_i): While the barycentric coordinate values define a "partition of unity," i.e., $\sum u_i = 1$, they do not satisfy "positivity," i.e., the condition that $u_i \geq 0$ (for all i) — better referred to as "non-negativity." Nevertheless, even though barycentric coordinate values can be negative, we can still use them as "quasi-probabilities": The figures (left and bottom) illustrate the concept of a point p belonging to a "multi-class $i_0 i_1 \ldots i_\ell$". The barycentric coordinates of p with close-to-zero values determine those classes that are NOT, or are barely, represented by the point p.



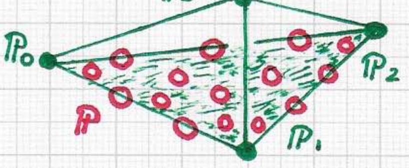
…



All points p have a barycentric coordinate tuple of the form $(u_0, u_1, 0, 0)$. Thus, while such points can be far away from p_0 and p_1, they belong to the interior of the 1-simplex with vertices p_0 and p_1. One can say that the points p represent the "multi-class 01" exactly: $u_0 + u_1 = 1$ and $u_0, u_1 \geq 0$.



Here, points p represent the "multi-class 01" only approximately: Points p lie in a small ϵ-neighborhood of the edge defined by the 1-subsimplex with vertices p_0 and p_1.



Points p belong to the "multi-class 012": $u_0 + u_1 + u_2 = 1$ and $u_0, u_1, u_2 \geq 0$.