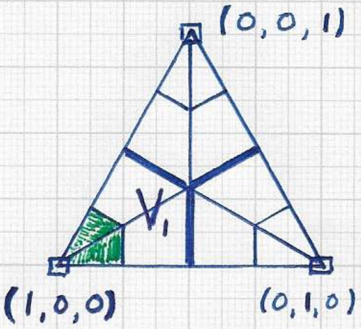
■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions: ... The figure (left) shows how one can down-scale an original Voronoi tile of the simplex. Down-scaling produces a "smaller copy" of the original tile, preserving the tile's shape. Thus, one must establish the conditions for an $\mathbb{m}$-tuple to define a point that resides inside such a down-scaled Voronoi tile. One can understand the general principles for these conditions by considering the **2**-simplex case, see figures (left). The geometry involved is simplified by the fact that the triangle's perpendicular edge bisectors are simultaneously also the triangle's angle bisectors. Concerning notation, we understand a Voronoi tile to be the union of two sub-regions, i.e.,

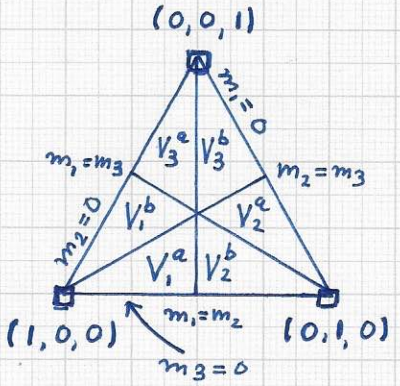
$$V_i = V_i^a \cup V_i^b \quad , \; i = 1,2,3 \, .$$

One can now determine the three conditions (inequalities) that a tuple $\mathbb{m}$ must satisfy to guarantee that the associated point lies inside a specific region V_i^a or V_i^b. Each region is bounded by segments of the **2**-simplex boundary and edge/angle bisectors. ...



Defining sub-region inside Voronoi tiles by down-scaling original tiles, using $\mathbb{m}$-tuples of "pure" materials – (1,0,0) etc. – as scaling centers.

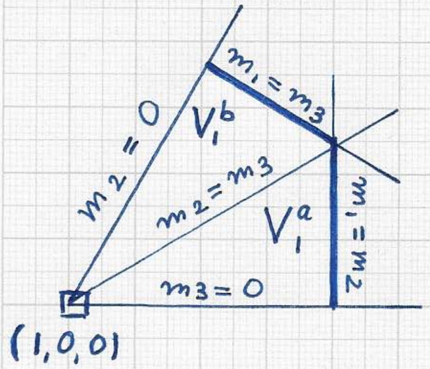
Note. The special simplex is characterized by the fact that the perpendicular edge bisectors are also the angle bisectors.



Definition of Voronoi tiles V_i as unions of two sub-regions, V_i^a and V_i^b. This definition is used to determine the conditions for a tuple $\mathbb{m}$ to be inside a down-scaled tile.

■ OBJECT AND MATERIAL EIGENFUNCTIONS - Cont'd.

• Laplacian eigenfunctions: ... The figure (left) shows only the geometry relevant for a tuple $\mathbb{m}$ to lie inside V_1^a or V_1^b, the two regions defining V_1. The tuple $\mathbb{m} = (m_1, m_2, m_3)$ must satisfy the following conditions to be inside:



Focusing on the two regions V_1^a and V_1^b that can be defined via three inequalities (each region).

V_1^a : $m_3 > 0 \wedge m_1 > m_2 \wedge m_2 > m_3$

V_1^b : $m_2 > 0 \wedge m_1 > m_3 \wedge m_3 > m_2$

Similarly, one obtains the inequality conditions for Voronoi tiles V_2 and V_3 and their respective sub-regions:

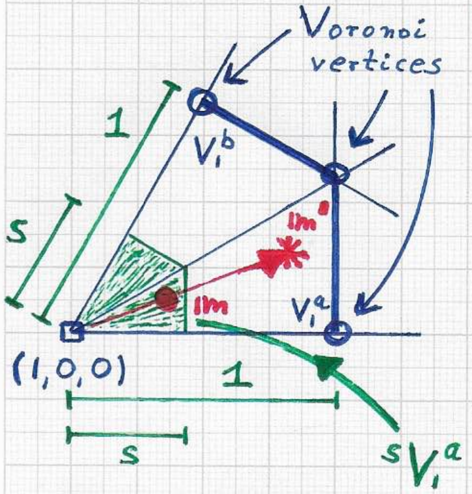
V_2^a : $m_1 > 0 \wedge m_2 > m_3 \wedge m_3 > m_1$

V_2^b : $m_3 > 0 \wedge m_2 > m_1 \wedge m_1 > m_3$

V_3^a : $m_2 > 0 \wedge m_3 > m_1 \wedge m_1 > m_2$

V_3^b : $m_1 > 0 \wedge m_3 > m_2 \wedge m_2 > m_1$

These conditions can also be used, in a slightly adapted way, to determine whether an $\mathbb{m}$-tuple defines a point inside a down-scaled Voronoi tile.



Test used to determine whether a tuple $\mathbb{m}$ lies inside the shaded region, i.e., the shown down-scaled version of Voronoi tile V_1. Tile V_1 was down-scaled by the scaling factor $s \in [0,1]$. "Tuple $\mathbb{m}$ is scaled" by the factor $1/s$, generating the tuple $\mathbb{m}'$. One can now test whether $\mathbb{m}'$ is inside V_1.

${}^sV_1^a$ = scaled-down version of V_1^a

• **TEST PRINCIPLE:** Voronoi tile V_1 has surrounding Voronoi vertices with a "normalized distance of 1" to $(1,0,0)$; V_1 is scaled-down by scaling factor s, with $0 \le s \le 1$, see figure (left, bottom). ...

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions:... Considering region V_1^a, we determine whether $\mathbf{m} = (m_1, m_2, m_3)$ is inside the scaled-down region ${}^sV_1^a$ by performing a scaling operation and subsequent test:

1) Compute the difference vector
$\mathbf{d} = \mathbf{m} - (1,0,0) = (m_1 - 1, m_2, m_3)$.

2) Scale this vector by $1/s$, producing the vector
$\mathbf{d}' = \frac{1}{s}\,\mathbf{d}$.

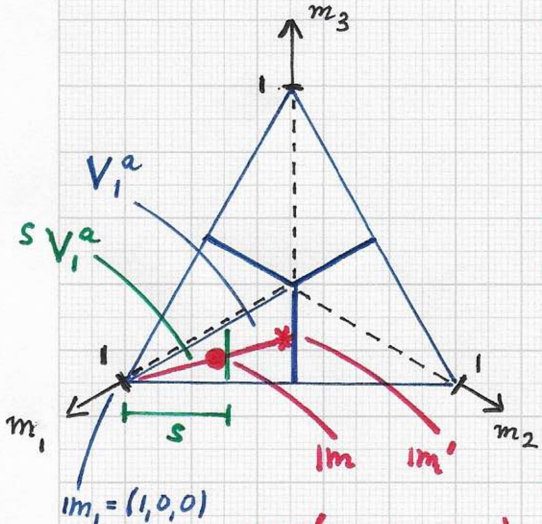
3) Add this scaled vector to the tuple $(1,0,0)$, creating
$\mathbf{m}' = \mathbf{m}_1 + \mathbf{d}' = (1,0,0) + \mathbf{d}'$
$= (m_1', m_2', m_3')$.

4) Concatenating these three transformations leads to

$$\underline{\underline{\mathbf{m}' = \left(1 + \frac{m_1 - 1}{s}, \frac{m_2}{s}, \frac{m_3}{s}\right)}}.$$

5) Perform test to determine whether $\mathbf{m}'$ lies inside V_1^a:
IS $(m_3' > 0 \wedge m_1' > m_2' \wedge m_2' > m_3')$ TRUE?
If it is true, $\mathbf{m}$ is inside the scaled-down version of V_1^a, i.e., ${}^sV_1^a$.

...



$\mathbf{m} = (m_1, m_2, m_3)$

$\mathbf{m}' = (m_1', m_2', m_3')$

Illustration of geometry and data involved in determining whether tuple $\mathbf{m}$ is inside a specific tile sub-region or not. The illustration shows a scaling factor s with value $3/5$. Using the mappings summarized on this page, one maps $\mathbf{m}$ to $\mathbf{m}'$ and tests whether $\mathbf{m}'$ satisfies the three conditions of a point to be inside V_1^a. If $\mathbf{m}'$ is in V_1^a, then $\mathbf{m}$ is in ${}^sV_1^a$.

• **Alternative method:** Knowing $\mathbf{m}$ and s, one could calculate the maximal scaling factor s_{max} that – when used in the summarized transformation – maps $\mathbf{m}$ to a point in the 2-simplex where $m_1 = m_2$ holds. If $s \leq s_{max}$, then $\mathbf{m}$ is inside ${}^sV_1^a$.

OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

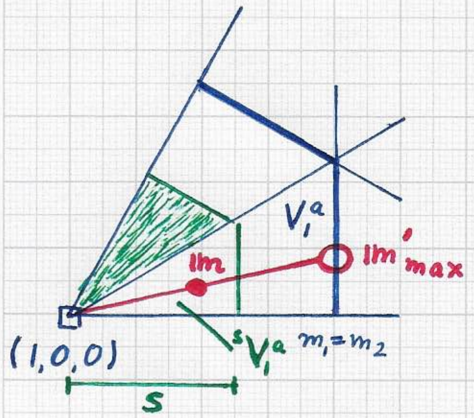
• **Laplacian eigenfunctions:**... The figure (left) shows the geometry involved in performing the **alternative test method** mentioned on the previous page. This alternative method calculates the maximal scaling factor s_{max} that would have to be used in the transformations summarized on the previous page to map **Im to Im'$_{max}$**, i.e., an Im'-tuple that lies on the boundary of V_1^a, where $m_1 = m_2$ **holds.** The general formula describing the mapping of Im to Im' is

$$Im' = \left(1 + \frac{m_1 - 1}{s}, \frac{m_2}{s}, \frac{m_3}{s}\right).$$

When Im' lies on the boundary of V_1^a, where $m_1 = m_2$, we obtain a special boundary tuple Im' = Im'$_{max}$ = **o**. Since components m_1' and m_2' must be equal for Im'$_{max}$, one obtains the condition

$$1 + \frac{m_1 - 1}{1/s_{max}} = \frac{m_2}{1/s_{max}}$$

$$\Rightarrow \quad s_{max} = (1 - m_1 + m_2)^{-1}$$



Alternative test for determining whether Im is inside $^sV_1^a$: Calculate a scaling factor s_{max} needed to map Im to a boundary location Im'$_{max}$. Then:

Im inside $^sV_1^a$

$$\Leftrightarrow \quad s \le s_{max}.$$

!

• Note. One can also derive the value of s_{max} by intersecting the parametric line

$$(1,0,0) + s_{max}(m_1 - 1, m_2, m_3)$$

with the implicit hyperplane

$$m_1 - m_2 = 0.$$

This leads to the equation

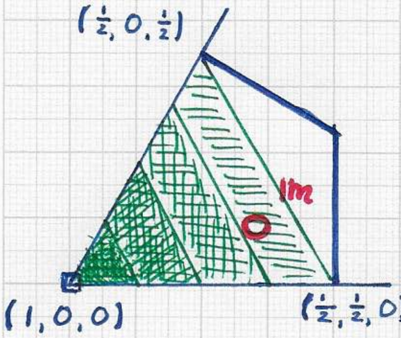
$$1 + s_{max}(m_1 - 1) - s_{max} m_2 = 0.$$

$$\Rightarrow s_{max} = \frac{1}{1 - m_1 + m_2}.$$

...

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions: ... The three discussed metrics to measure "distance to the pure material tuple (1, 0, 0)" make it possible to define **"quasi-probabilities" p** to m-tuples: The value of p should be 1 for $\mathbb{m} = (1,0,0)$ and should decrease to 0 with increasing distance to (1, 0, 0).

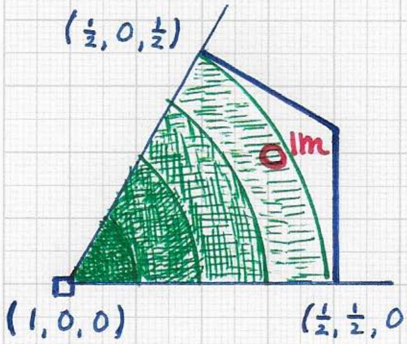


Method 1 to associate a probability p-value with tuple m. The p-value magnitude is represented by the used shading intensity.

Method 1 to define a p-value for $\mathbb{m} = (m_1, m_2, m_3)$ is shown in the left figure (top):

$$p = \begin{cases} 1 - 2m_1 & , \ \frac{1}{2} \le m_1 \le 1 \\ 0 & , \ m_1 > \frac{1}{2} \end{cases} .$$

(The condition(s) for m_i-values concern the interior of the 2-simplex.)

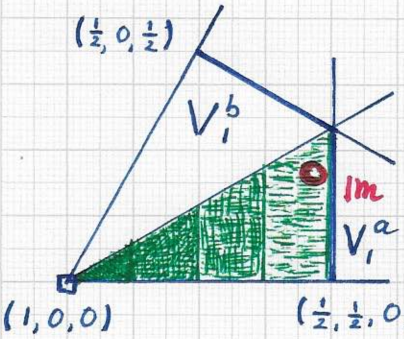


Method 2 to associate a p-value with m. Shading intensity represents p-value magnitude.

The middle figure illustrates the p-value magnitude variation of method 2:

$$p = \begin{cases} 1 - \sqrt{2}\, r & , \ 0 \le m_2, m_3 \le \frac{1}{2} \\ & \ \ \wedge \ m_2 + m_3 \le \frac{1}{2} \\ 0 & , \ \text{otherwise} \end{cases} ,$$

where $r = \sqrt{2}\,(m_2^2 + m_2 m_3 + m_3^2)^{1/2}$.



Method 3 to associate a p-value with m in sub-region V_1^a.

The bottom figure shows p-value variation generated via method 3:

$$p = \begin{cases} m_1 - m_2 & , \ \mathbb{m} \text{ inside } V_1^a \\ 0 & , \ \text{otherwise} \end{cases} .$$

...