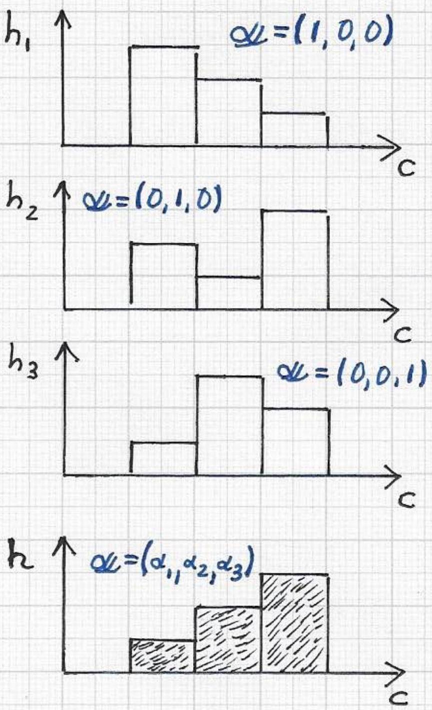
■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions: ... By interpreting the distance value of an m-tuple as a quasi-probability for m to be of a "pure" material class $(1, 0, \ldots, 0), \ldots$ or $(0, \ldots, 0, 1)$, it is possible to devise a **PROBABILITY-BASED CLASSIFICATION METHOD.** The basic concepts and necessary computations can be explained via a low-dimensional example; again, we use the 2-simplex and 3 "pure" material classes. We tackle the multi-class, multi-scale material classification problem, where multi-scale data properties are captured via an eigenfunction-based local data representation and multiple classified data sets are known for each class (class samples / segments). Given a new, unclassified datum – an m-tuple – one can compute a quasi-probability for a given scale-specific m-tuple of an unclassified datum and calcute probability values for "$P_{cl,sg,sc}$" of class-segment-scale-agreement with classified data. ...



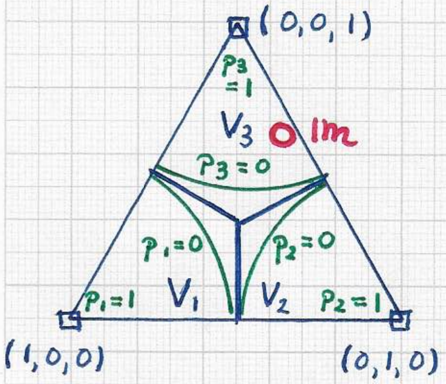
A histogram h (shaded) is given for an unclassified data set; h reflects the normalized distribution of coefficient (c) values resulting from a multi-scale eigenfunction-based analysis of a new segment. The histograms h_1, h_2 and h_3 capture the same type of coefficient information for 3 sample segments of the same class. The number of bins for coefficient values is 3 as well. Thus, one can uniquely compute

$$h = \sum_{i=1}^{3} \alpha_i h_i(c).$$

All four histograms can now be identified with a tuple $(\alpha_1, \alpha_2, \alpha_3) = \alpha$.

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions: … We consider the 2-simplex shown in the figure (left). The figure illustrates the three "radial distance" probabilities p_1, p_2 and p_3 associated with tuples $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$, respectively. The isolines $p_1=0$, $p_2=0$ and $p_3=0$ are sketched via circular arcs.



Simple illustration of "probability functions" p_1, p_2 and p_3. These functions vary between 0 and 1 in Voronoi tiles V_1, V_2 and V_3. For a given tuple $\mathbb{m}$ one can now compute a "quasi-probability" for $\mathbb{m}$ to be of class 1, 2 or 3.

The linearly varying probability values in Voronoi tiles V_1, V_2 and V_3 are

$$\underline{\underline{p_{i1}}} = \begin{cases} 1 - \sqrt{2}\, r_{i1}, & 0 \le m_{i2}, m_{i3} \le \frac{1}{2} \\ & \wedge\ m_{i2} + m_{i3} \le \frac{1}{2} \\ 0, & \text{otherwise} \end{cases},$$

where $r_{i1} = \sqrt{2}\left(m_{i2}^2 + m_{i2} m_{i3} + m_{i3}^2\right)^{1/2}$

and $i1, i2, i3 \in \{1, 2, 3\}$, $i1 \neq i2$, $i1 \neq i3$, $i2 \neq i3$.

The unclassified tuple is the coefficient tuple $\boldsymbol{\alpha}$, i.e.,

$$\mathbb{m} = \boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3).$$

• The sketches of the histogram functions $h_1(c)$, $h_2(c)$ and $h_3(c)$ – previous page – show that the actual positional vectors of these functions define 3 points on the 2-sphere in 3D space that have general coordinate tuples

$$\boldsymbol{\alpha}_i = (\alpha_1^i, \alpha_2^i, \alpha_3^i),$$

$i = 1, 2, 3$. One must consider this in all computations. **!**

• **Important Notes:** **!**

The normalized "histogram positional vectors" for $h_1(c)$, $h_2(c)$ and $h_3(c)$ define points on the unit 2-sphere in 3D space. The same is true for $h(c)$ – and one must use one of the described methods to map the point identified with $h(c)$ to the 2-plane defined by the points for h_1, h_2 and h_3.

…

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions: ...

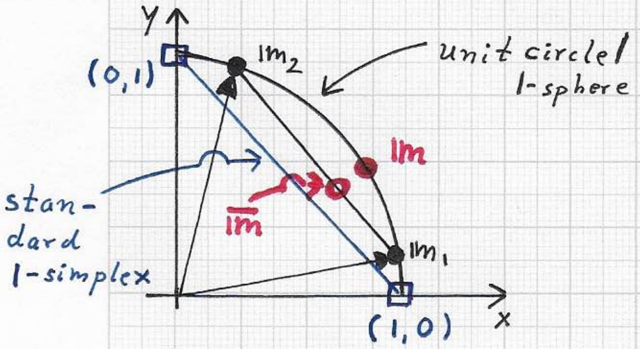


Illustration of the necessary mappings to be performed in a pre-processing step. The goal is to calculate a probability value for an answer to the question: "Is tuple im of class im$_1$ or class im$_2$?"

im$_1$, im$_2$: tuples/points of class 1 and class 2 (or of the same class represented via 2 samples)

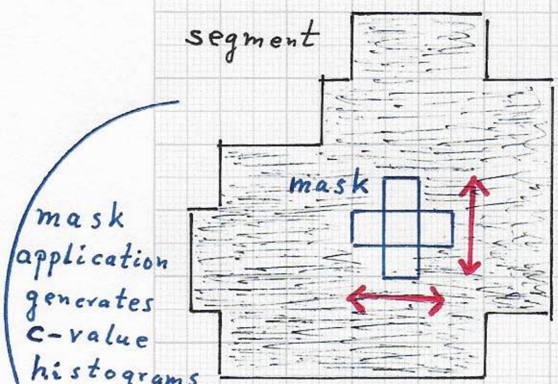
im, $\overline{\text{im}}$: tuple/point to be classified; im represents the given normalized histogram function, $\overline{\text{im}}$ represents the result of mapping im onto/into the 1-simplex having im$_1$, im$_2$ as end points

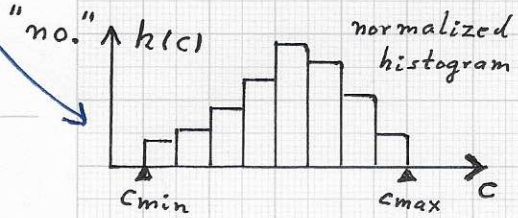
(1,0), (0,1): coordinate tuples of the vertices of the "STANDARD 1-SIMPLEX" in the 2D plane

The figure (left) summarizes the important notes from the previous page. They concern the issues of working with NORMALIZED HISTOGRAM FUNCTIONS/DATA and the use of BARYCENTRIC COORDINATES relative to a so-called STANDARD (k-1)-SIMPLEX IN k-DIMENSIONAL SPACE. To summarize the basic problem, we have stored histogram functions/data (normalized) for classified material samples (im$_1$, im$_2$). One must determine a probability for a new, unclassified histogram function/data (also normalized), im, based on a distance measure for the one unclassified and the classified samples. First, in order to "represent im barycentrically" with respect to a 1-simplex, we map im to $\overline{\text{im}}$, where $\overline{\text{im}}$ lies on the line (segment) through im$_1$ and im$_2$ (example in figure). Second, one must perform a linear transformation that maps im$_1$ (im$_2$) to (1,0) ((0,1)), to be able to perform probability estimation in a STANDARD SIMPLEX. ...

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions: ... It is important to recall the actual meaning of the IM-tuples and the underlying histogram functions/data: Each histogram function describes the DISTRIBUTION OF COEFFICIENT VALUES THAT RESULT FROM ANALYZING A DATA SEGMENT IN A MULTI-SCALE FASHION VIA LAPLACIAN EIGENFUNCTIONS. Thus, the overarching classification goal requires us to compare scale-specific coefficient value distributions and calculate a probability for a "match" between an unclassified **segment** and a classified sample(s). The figure (left) shows the data analysis and classification context: A convolution mask applied to an image data segment throughout produces the coefficients of many local eigenfunction-based image data expansions. Once an entire segment has been analyzed in this multi-scale manner, one has histograms of coefficient value distributions. Using B bins to discretize a $[c_{min}, c_{max}]$ value range, one obtains a "B-dimensional IM-tuple." ...





Characterizing a segment via histogram(s) of Laplacian eigenfunction coefficient values. A segment is analyzed by a mask that represents the segment's pixel data values covered by the mask in a multi-scale Laplacian eigenfunction basis. The mask traverses the entire segment and generates, for each mask location, a tuple of coefficient (c) values, the local coefficients of the mask's underlying eigenfunctions. Having traversed (convolved) the entire segment, one has coefficient value distributions $h(c)$ for $c \in [c_{min}, c_{max}]$ – for "low-, medium- and high-frequency" eigenfunctions.

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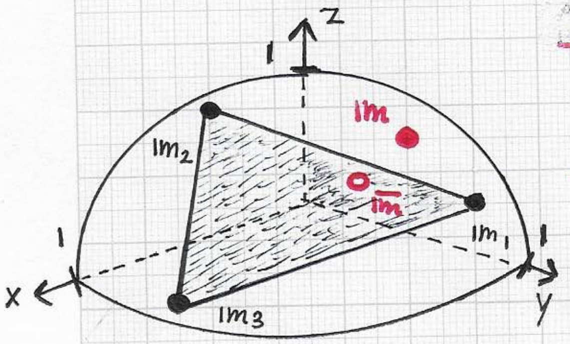
■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions: … For example, if one had k sample segments and one were to represent each segment's normalized coefficient value histogram with B = k bins, one could use these histograms (= piecewise constant histogram functions) as a basis to compute the expansion of an unclassified histogram. Alternatively, one can employ a barycentric coordinate-based method to represent the unclassified histogram relative to a (k-1)-simplex having the k classified sample histograms as its vertices.



Normalized coefficient value histogram functions (k = B bins) become B-dimensional tuples im_i; these tuples are represented in the figure as points im_1, im_2 and im_3. The unclassified datum is im, and in order to express this datum "meaningfully" with respect to the 2-simplex with vertices im_1, im_2 and im_3, one must map im to $\overline{im}$, i.e., a point in the plane of the triangle.

First, it is assumed that the approximation $\overline{im}$ is a "good approximation of im in the sense that it makes possible the definition and computation of a measure for distance to the classified tuples/points im_1, im_2 and im_3 in the 2-simplex.

Second, in order to perform distance computations relative to the STANDARD 2-SIMPLEX, one linearly maps $\overline{im}$, im_1, im_2 and im_3 – where $im_1 \mapsto (1,0,0), \dots, im_3 \mapsto (0,0,1)$.

• **Note.** In the general setting, a specific material class can be represented by only ONE or MORE THAN ONE classified samples/sample histograms.

For the purpose of the subsequent discussion, we want to represent the tuple/point $\overline{im}$ barycentrically in the STANDARD 2-SIMPLEX with vertices (1,0,0), (0,1,0) and (0,0,1). Thus, $\overline{im}$, im_1, im_2 and im_3 must first be linearly mapped. …