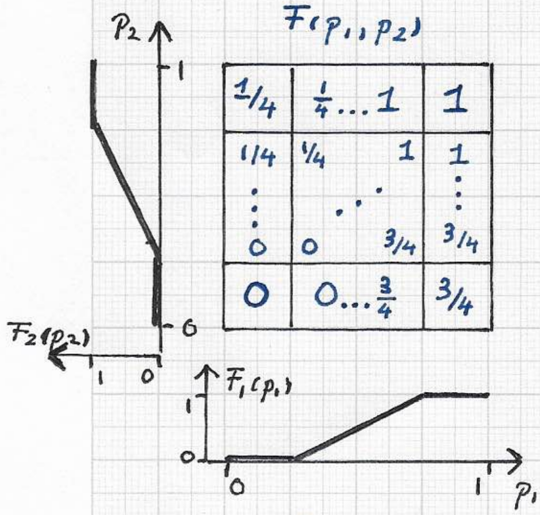
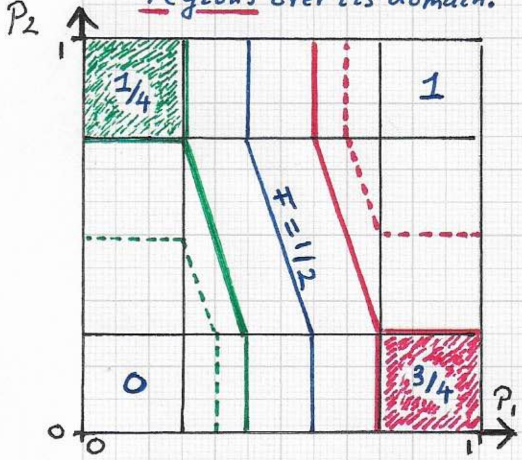
OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

- *Laplacian eigenfunctions and neural networks:...*



$F(p_1,p_2) = \frac{3}{4} F_1(p_1) + \frac{1}{4} F_2(p_2)$

F_1 and F_2 are **"BLaC"** (blending-of-linear-and-constant) functions. Thus, a convex combination F is also a (bivariate) "BLaC" function, having constant plateaus and linear regions over its domain.



Contours of $F(p_1,p_2)$:

F = 1/8 1/4 2/4 3/4 7/8

The figures shown on the previous two pages use two univariate decider functions F_1 and F_2 with simple Linear "RAMP" functions for their interior intervals. The three convex combinations illustrated in those figures sketch the contour/isoline behavior and the classification regions in the (p_1, p_2)-domain where $F(p_1,p_2) > 3/4$. **It is evident that the used weights in the convex combination have substantial impact on the classification regions and thus classification performance.** In this simple example, the degrees of freedom to be optimized are the values of w_i in $w_1 F_1 + w_2 F_2$ and the value of the threshold T in $F(p_1,p_2) > T$, defining the boundary of the classification region.

The convex combination shown in the left figures, $F = 3/4 F_1 + 1/4 F_2$, combining simple **"BLaC"** functions F_1 and F_2, demonstrates that its contours $F = T$, for $T \in \{1/8, 1/4, 1/2, 3/4, 7/8\}$, define simple yet very different classification regions. ...

OBJECT AND MATERIAL EIGENFUNCTIONS — Cont'd.

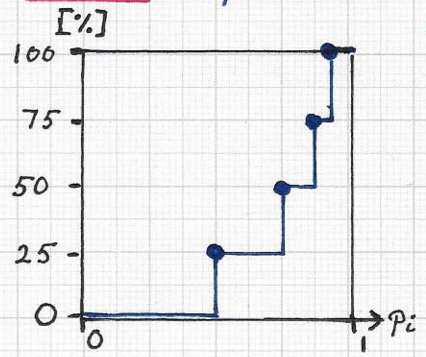
- **Laplacian eigenfunctions and neural networks: ...**

Definition of individual univariate decider function $F_i(p_i)$ via "training with material class samples":

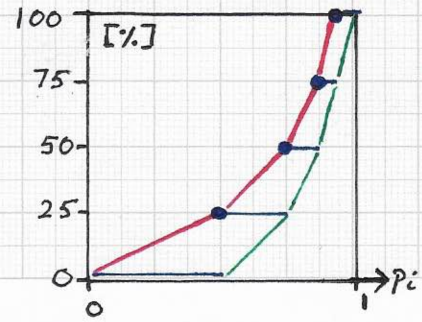
i) Distribution of p-values for p_i (scale i) resulting from four samples of a specific material class on the p_i-line:

The p_i-values for the samples are $1/2, 3/4, 7/8, 15/16$.

ii) Cumulative distribution step function:



Step function's envelopes:



The envelopes are piecewise linear.

Since $F(p_1, p_2)$ is a bivariate "BLaC" function, four of its contours include 2D regions in its (p_1, p_2)-domain; these contours are the contours $F = 0$, $F = 1/4$, $F = 3/4$, $F = 1$. For these threshold / contour values F's graph contains "plateaus." This behavior of F does not present a problem when using F for classification purposes: it is only necessary to evaluate F effectively and efficiently for a given value of (p_1, p_2).

THOUGHTS CONCERNING THE "DIRECT AND OPTIMAL COMPUTATION" OF THE INDIVIDUAL DECIDER FUNCTIONS $F_i(p_i)$, $i = 1 \ldots H$, AND THE OVERALL DECIDER FUNCTION $F(p_1, \ldots, p_H)$ WHEN DEFINED AS A CONVEX COMBINATION $F = \Sigma w_i F_i$:

- **Definition of $F_i(p_i)$:** The values of F_i's inner knots, i.e., p_1^i and p_2^i, define the "RAMP" interval; these values could be 'learned optimally' from samples.

(See left figures.)

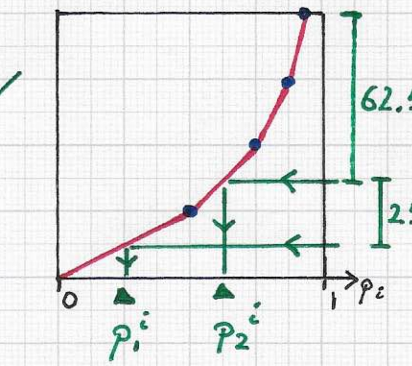
...

Stratovan

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks: ...

iii) Definition of inner knot values via 'left envelope':



iv) Resulting decider function $F_i(p_i)$:

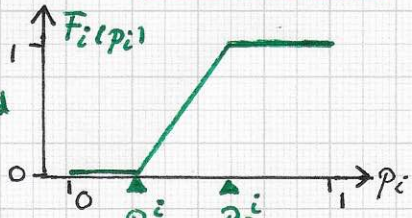
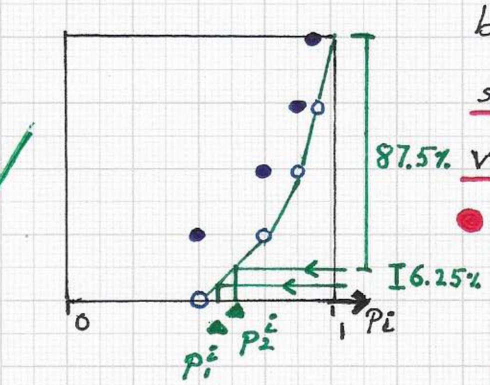


Illustration of the final BLaC decider function $F_i(p_i)$



Alternative definition of inner knot values via 'right envelope'

Considering the example shown in the figures (previous page and this page), one could use the 'left envelope,' shown in red, to define the values of the inner knots p_1^i and p_2^i: For example, one could require that the 'top 62.5% of the left envelope' define the interval $[p_2^i, 1]$; the figure (left) illustrates how the value of p_2^i is computed. Similarly, one could require that the "RAMP" be defined 'by the next 25%' of the sample data, as implied by the left envelope; the figure (left, top) shows how the value of p_1^i is calculated. Thus, the "RAMP" interval (p_1^i, p_2^i) is established. This procedure is fully automated – but it is necessary to provide the sample data and the two percentage values (62.5% and 25% in the example).

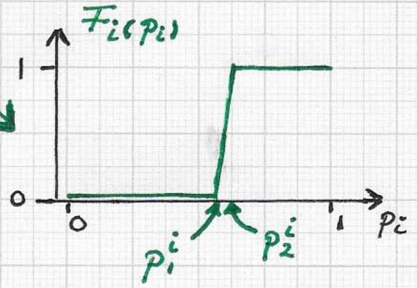
● **Definition of $F(p_1, \ldots, p_H)$:**

Considering the bivariate case ($H=2$), F is the convex combination

$$F(p_1, p_2) = w_1 F_1(p_1) + w_2 F_2(p_2)$$
$$= w_1 F_1(p_1) + (1-w_1) F_2(p_2)$$
$$= w_1 \left(F_1(p_1) - F_2(p_2)\right) + F_2(p_2) \quad \ldots$$

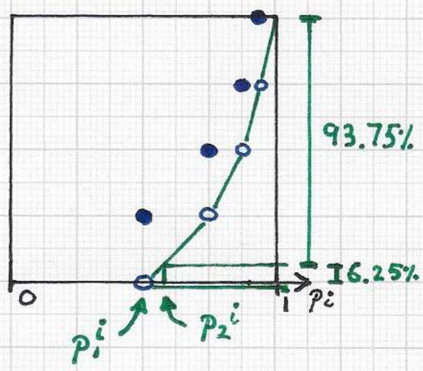
OBJECT AND MATERIAL EIGENFUNCTIONS - Cont'd.

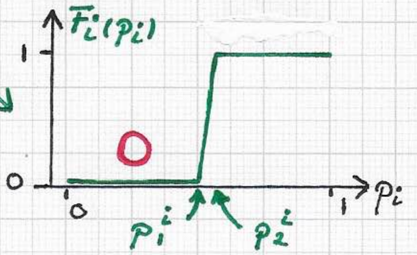
• Laplacian eigenfunctions and neural networks: ...



Alternative final BLaC function $F_i(p_i)$ with much steeper "RAMP"

Extreme choice of inner knot value p_1^i: choose location of the given sample with minimal p_i-value:





This choice of the value of p_1^i maximizes the 0-interval of $F_i(p_i)$.

⇒ IF THE EXACT, TRUE BIVARIATE DISTRIBUTION/DENSITY FUNCTION $\varrho(p_1,p_2)$ IS KNOWN, THEN ONE WILL BE ABLE TO COMPUTE THE VALUE OF w_1 — the one degree of freedom in the bivariate case — OPTIMALLY. Given the two individual univariate decider functions $F_1(p_1)$ and $F_2(p_2)$, the optimal w_1-value can be computed as the least-squares solution to the minimization problem $(F(p_1,p_2) - \varrho(p_1,p_2))^2 \rightarrow \min.$

!

Thus, one uses the least-squares method to compute the optimal, best approximation to the problem

$$(F_1(p_1) - F_2(p_2))\, w_1 = \varrho(p_1,p_2) - F_2(p_2).$$

The best approximation is defined by

$$\langle F_1 - F_2, F_1 - F_2 \rangle\, w_1 = \langle \varrho - F_2, F_1 - F_2 \rangle ,$$

where the inner product of f and g is

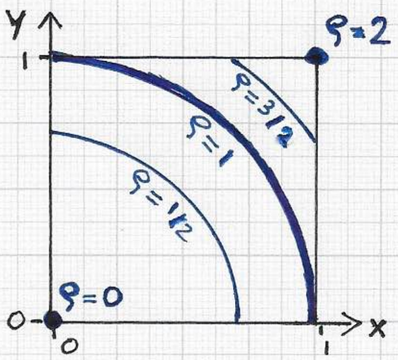
$$\langle f, g \rangle = \int_{p_2=0}^{1} \int_{p_1=0}^{1} f(p_1,p_2)\, g(p_1,p_2)\, dp_1\, dp_2.$$

The resulting w_1-value is optimal.

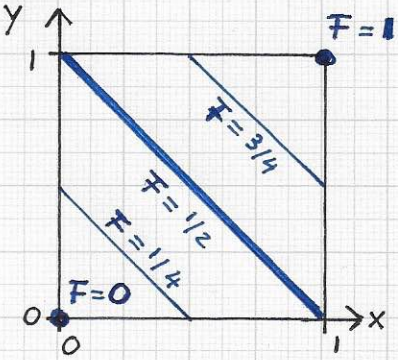
...

OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks:...



Contours of $\rho = x^2+y^2$



Contours of $F = (x+y)/2$

The sketched contours show that F indeed approximates ρ well.

Using the "standard" basis $\{1, x, y\}$ to compute a best approximation $L(x,y) = c_0 + c_1 x + c_2 y$ leads to a different result.

• Example. Given the two univariate functions $F_1(x) = x$ and $F_2(y) = y$, compute the optimal convex combination $F(x,y) = w\,F_1(x) + (1-w)\,F_2(y)$ that is the best approximation to the function $\rho(x,y) = x^2+y^2$, $x, y \in [0,1]$, in the least-squares error metric!

➜ Problem: $(F_1 - F_2)\,w = \rho - F_2$

$$\Leftrightarrow (x-y)\,w = x^2+y^2-y$$

➜ Solution: $\langle x-y, x-y\rangle\, w = \langle x^2+y^2-y, x-y\rangle$

$$\Leftrightarrow \int_0^1\!\!\int_0^1 (x-y)^2\,dx\,dy \;\; w = \int_0^1\!\!\int_0^1 (x^2+y^2-y)(x-y)\,dx\,dy$$

$$\Leftrightarrow \frac{1}{6}\,w = \frac{1}{12}$$

$$\Leftrightarrow w = \frac{1}{2}$$

Thus, the optimal approximating convex combination is $F(x,y) = \frac{1}{2}x + \frac{1}{2}y$. The sketches shown in the figures (left) illustrate that contours of F and ρ have 'similar qualitative behavior' and that ρ's function values over the unit square are optimally approximated.

• **Note.** When giving up the convex combination requirement and calculating the best approximation as a linear polynomial $L(x,y) = c_1 x + c_2 y$ or $L(x,y) = c_0 + c_1 x + c_2 y$, different solutions result.

...