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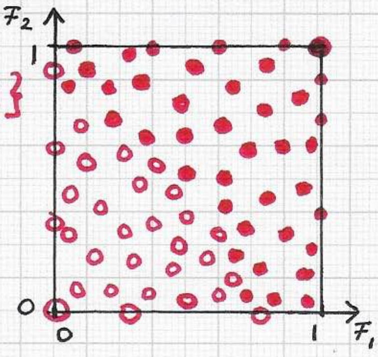
■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks:...

Given a set of samples $\mathbb{F}_i = (F_1^i, F_2^i)$, $i = 1 \dots N$, with exact, known, defined classification values $F_i = F(\mathbb{F}_i)$, one can use the least-squares method to construct an optimal function $F(\mathbb{F})$ as a convex combination:

$\{(F_1^i, F_2^i)\}$ and $\{F_i\}$



• **Here: Classification values are 0 (○) or 1 (●).**

$$F(\mathbb{F}) = \sum_{sc=1}^{H} w_{sc} F_{sc}$$

• **Conditions:**

$$F_i = \sum_{sc=1}^{H} w_{sc} F_{sc}^i, \quad i = 1 \dots N$$

• **H = 2 (figure):**

$$F_1 = w F_1^1 + (1-w) F_2^1$$
$$\vdots$$
$$F_N = w F_1^N + (1-w) F_2^N$$

$$\begin{bmatrix} F_1 - F_2^1 \\ \vdots \\ F_N - F_2^N \end{bmatrix} = \begin{bmatrix} F_1^1 - F_2^1 \\ \vdots \\ F_1^N - F_2^N \end{bmatrix} w$$

$$\mathbb{b} = M \quad \mathbb{w}$$

$$\Rightarrow \mathbb{w} = (M^T M)^{-1} M^T \mathbb{b}$$

Once again, this is an appropriate stage of the discussion to reflect and be clear about "meaning" and the "semantic context" of the individual mathematical concepts and methods employed. First, we must consider proper ways to 'combine' the univariate decider functions $F_{sc}(p_{sc})$ of all the scales sc (of a specific class). A simple convex combination $F = \sum w_{sc} F_{sc}(p_{sc})$ uses constant weights w_{sc}, where $\sum w_{sc} = 1$ and $w_{sc} \geq 0$. A more general form of a convex combination would use, for example, weight functions $w_{sc} = w_{sc}(F_1, \dots, F_H)$, where $\sum w_{sc}(F_1, \dots, F_H) = 1$ (partition of unity) and $w_{sc}(F_1, \dots, F_H) \geq 0$ (non-negativity). The values of constant weights w_{sc} can be computed optimally via the least-squares method (left). Non-negativity of the weight values is directly included in the equation system as a constraint $w_H = 1 - (w_1 + \dots + w_{H-1})$. In the 2D case, $w = w_2 = 1 - w_1$, for example.

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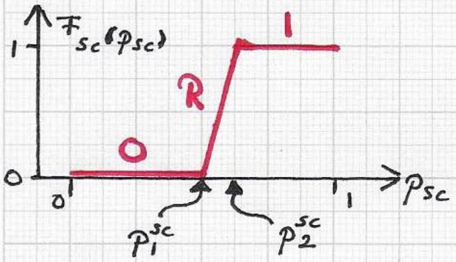
• Laplacian eigenfunctions and neural networks:...

"Meaning" of a univariate decider function $F_{sc}(p_{sc})$:



One makes a match or mis-match decision purely based on F_{sc}:

$F_{sc} = 0 \Rightarrow$ segment is a definite mis-match

$F_{sc} = 1 \Rightarrow$ segment is a definite match

$F_{sc} = R(p_{sc}) \Rightarrow$ segment could be a match or mis-match

• A 'CORRECT DECISION' CONCERNING MATCH (1) OR MIS-MATCH (0) SHOULD/MUST BE DONE IN CONJUNCTION WITH ALL OTHER F_{sc} FUNCTIONS, AND 'CONTRADICTIONS' MUST NOT ARISE!

The least-squares best approximation example provided on the previous page used **H = 2**. We briefly summarize the corresponding equations for the **case H=3 to** see the pattern for the general case.

• $H = 3 \Rightarrow w_3 = 1 - w_1 - w_2$

• $F_1 = w_1 F_1^1 + w_2 F_2^1 + (1 - w_1 - w_2) F_3^1$

$\vdots$

$F_N = w_1 F_1^N + w_2 F_2^N + (1 - w_1 - w_2) F_3^N$

$$\begin{bmatrix} F_1^1 - F_3^1 & F_2^1 - F_3^1 \\ \vdots & \vdots \\ F_1^N - F_3^N & F_2^N - F_3^N \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} F_1 - F_3^1 \\ \vdots \\ F_N - F_3^N \end{bmatrix}$$

$\Rightarrow$ general case:

$$\underbrace{\begin{bmatrix} F_1^1 - F_H^1 & \dots & F_{H-1}^1 - F_H^1 \\ \vdots & & \vdots \\ F_1^N - F_H^N & \dots & F_{H-1}^N - F_H^N \end{bmatrix}}_{M} \underbrace{\begin{bmatrix} w_1 \\ \vdots \\ w_{H-1} \end{bmatrix}}_{\mathbb{w}} = \underbrace{\begin{bmatrix} F_1 - F_H^1 \\ \vdots \\ F_N - F_H^N \end{bmatrix}}_{\mathbb{b}}$$

$$\Rightarrow \quad \mathbb{w} = (M^T M)^{-1} M^T \mathbb{b},$$

$$w_H = 1 - w_1 - \dots - w_{H-1}.$$

For increasingly larger and larger values of **N** (i.e., $N \gg H$) approximation quality deteriorates.

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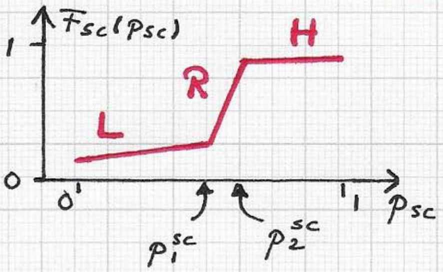
OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks:...

Changed definition of decider function $F_{sc}(p_{sc})$:



Definition of $F_{sc}(p_{sc})$ that is monotonically increasing and does not have "constant" plateaus." This function does not take on the values **0** and **1**. The function instead has three distinct regions of behavior:

$[0, p_1^{sc}]$: LOW (L)

(p_1^{sc}, p_2^{sc}): RAMP (R)

$[p_2^{sc}, 1]$: HIGH (H)

The "meaning" is:

L: segment is a mis-match (high probability)

H: segment is a match (high probability)

R: "RAMP" (as used before)

The left side of the previous page reviews the meaning of a univariate decider function $F_{sc}(p_{sc})$ as defined. It is desirable – and necessary – to alter the definition of $F_{sc}(p_{sc})$ slightly, to make possible the combination of the functions F_{sc}, sc = 1...H, into one overall decider function $F(F_1, ..., F_H)$ without contradictions – i.e., logical contradictions between "match" and "mis-match." In order to achieve this goal **one must not use constant-0 and constant-1 regions for the definition of a univariate decider function $F_{sc}(p_{sc})$.** The figure (left) provides an example. For the specific material class and scale being considered, **the value of "match probability" of a segment is a VERY LOW (L) value, a VERY HIGH (H) value or an R-value in the "RAMP" region. The individual functions F_{sc} can now be combined without contradictions for the construction of F.**

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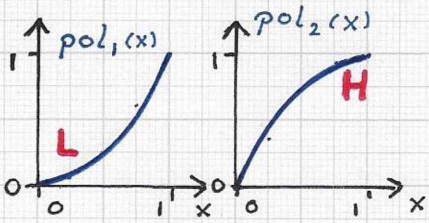
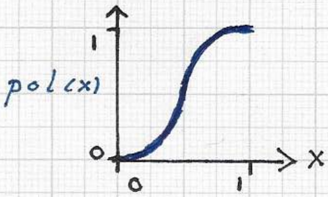
OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks:...

General idea of linearly blending two polynomial functions – the first polynomial having **LOW (L)** values close to 0, the second polynomial having **HIGH (H)** values close to 1:





• Example:

$pol_1(x) = x^2$

$pol_2(x) = (1-(x-1)^2)$

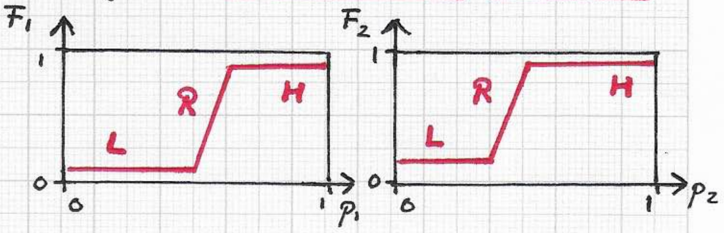
$pol(x) = (1-x)\, pol_1(x) + x\, pol_2(x) = -2x^3 + 3x^2$

This method for blending two the "two types of behavior" – LOW close to $x=0$ and HIGH close to $x=1$ is a method one can utilize when one combines univariate decider functions to define a multi-variate decider function $F(\mathbb{F})$.

One can state the guiding principle for the construction of an overall decider function $F = F(\mathbb{F}) = F(F_1, F_2)$, for example, as follows:

When combining a LOW (HIGH) value of F_1 with a LOW (HIGH) value of F_2, the resulting value should be even LOWER (HIGHER). When combining values of F_1 and F_2 from other pairs of behavioral regions, the resulting value should be a "meaningful blend."

For example, one must construct the function $F(F_1(p_1), F_2(p_2))$ from these two given univariate functions:



These two functions with their L-, R- and H-regions must be combined via an operation/operator that follows the guiding principle described above to construct $F(F_1, F_2)$.

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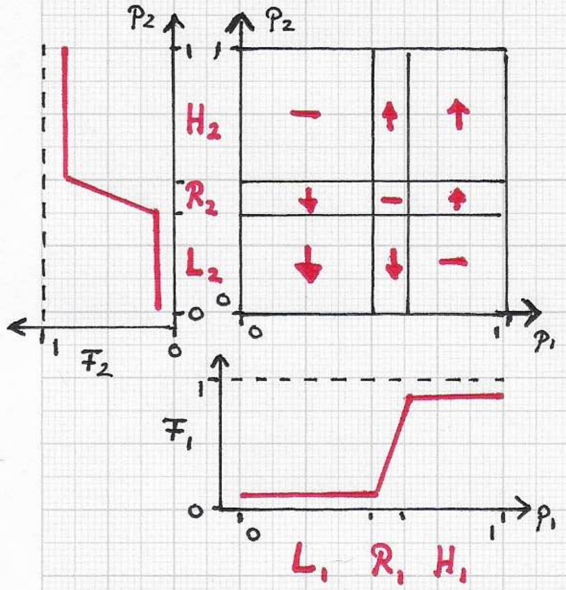
■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd

• Laplacian eigenfunctions and neural networks: ...

Construction of $F(F_1, F_2)$:



Abstract schematic illustration of the desired impact of combining all possible "type pairs" of region types of F_1 and F_2.

Function values from the L_1-, R_1- and H_1-regions of F_1 are combined with function values from the L_2-, R_2- and H_2-regions of F_2.

The 9 sub-domains within the unit domain square belong to 3 categories:

↓ "combination moves F-values even closer to 0"

↑ "combination moves F-values even closer to 1"

– "combination of F_1- and F_2-values has an averaging effect"

• **Note.** By ensuring that the univariate decider functions $F_{sc}(p_{sc})$ do NOT produce / contain value-0 or value-1 plateaus (or simply do NOT produce the values 0 or 1) one effectively avoids having to worry about the restrictions imposed by Boolean logic. (By viewing the numerical value 0 as FALSE and the value 1 as TRUE one must take care of not introducing or not recognizing logical contradictions in the decision-making and classification process.) The exclusive use of values in the open interval (0,1) as allowed values of the functions $F_{sc}(p_{sc})$ makes it possible to perform computations in the much less restricted setting of **FUZZY LOGIC**. The figure (left) explains at a rather abstract level how, for example, two LOW values of F_1 and F_2 should be combined to produce an "even lower value" for F: ↓

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