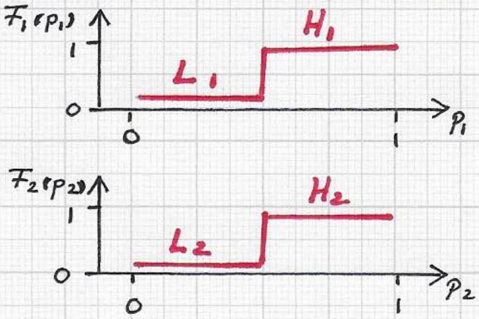
Stratovan

OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

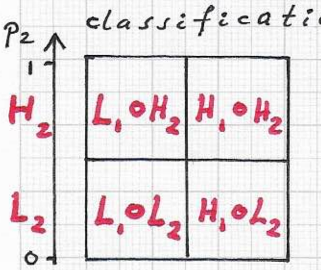
- Laplacian eigenfunctions and neural networks:...

Abstract view of two "idealized" univariate decider functions F_1 and F_2:



The two decider functions $F_1(p_1)$ and $F_2(p_2)$ have low and high values – L_1 and H_1, and L_2 and H_2, respectively.

The needed overall bivariate decider function $F(p_1, p_2)$ combines low (L) and high (H) values in a quasi-binary, quasi-Boolean manner – while considering the focus on and restrictions of data classification:



Abstract sketch of the four possibilities to combine the "regions" of F_1 and F_2.

This figure uses the symbol '$\circ$' for combining low and high values of F_1 and F_2.

The operator '$\circ$' must produce only result values that are viable, plausible for classification.

In the context of our overarching application – material/object classification – the best approximation method must be seen in the context of this application: We are **NOT** tackling a "typical function" approximation problem; we are concerned with "logical functions," "Boolean functions," "quasi-binary functions" used for classification, decision-making. Thus, we must keep in mind this application context and the implied combinatorial aspects when adapting and specializing a best approximation approach for a **"TRUE-FALSE" ("1-0")** setting.

We consider combinatorial aspects for the two-scale/2D case first. We must understand the pairs **"TRUE-FALSE," "T-F," "1-0,"** **"yes-no"** and **"high-low" ("H-L")** as synonyms in this discussion. The left figures illustrate the purely abstract view of the construction of decider function $F(p_1, p_2)$.

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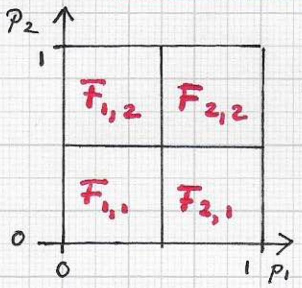
High-level, abstract view of the value behavior of the decider function $F(p_1,p_2)$:



The value of each "$F_{i,j}$ region" can be 0 or 1 – from a purely combinatorial view.

Combinatorial possibilities:

		$F_{1,1}$	$F_{2,1}$	$F_{1,2}$	$F_{2,2}$
	0	0	0	0	0
*	1	0	0	0	1
	2	0	0	1	0
*	3	0	0	1	1
	4	0	1	0	0
*	5	0	1	0	1
	6	0	1	1	0
*	7	0	1	1	1
	8	1	0	0	0
	9	1	0	0	1
	10	1	0	1	0
	11	1	0	1	1
	12	1	1	0	0
	13	1	1	0	1
	14	1	1	1	0
	15	1	1	1	1

The number of combinatorial possibilities is $2^4=16$. Only $2^2=4$ of these possibilities are viable for classification purposes, indicated by "*".

* Viability: $F_{1,1}$ must be 0 and $F_{2,2}$ must be 1. $\Rightarrow 2^2=4$ viable combination results.

!

In the two-scale/2D setting the combination operator '$\circ$' can only produce these viable results:

$\circ$ \ $F=$	F_1	F_2	$F_1 \vee F_2$	$F_1 \wedge F_2$
$L_1 \circ L_2$	L_1	L_2	L	L
$L_1 \circ H_2$	L_1	H_2	H	L
$H_1 \circ L_2$	H_1	L_2	H	L
$H_1 \circ H_2$	H_1	H_2	H	H

Using the truth/Boolean values F and T, or 0 and 1, one can also summarize the '$\circ$' results as:

$\circ$	$F=$
$L_1 \circ L_2$	0
$L_1 \circ H_2$	0 or 1
$H_1 \circ L_2$	0 or 1
$H_1 \circ H_2$	1

The classification context demands that the combination of only low (high) values always yields the value 0 (1).

The "physical justification" for these values of F is this:

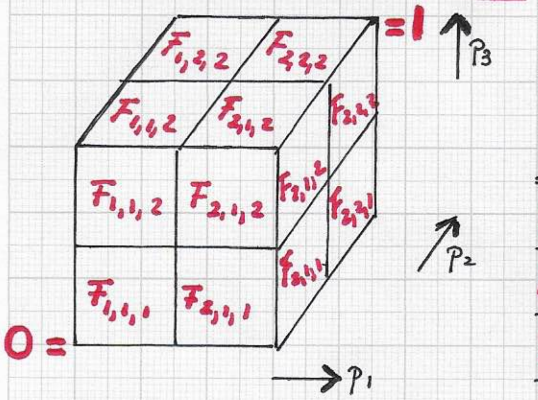
i) If all univariate decider function values are low (high), then the function value of F must be 0(1).

ii) If the univariate decider function values to be combined are not all low (high), then F's value can be 0 or 1.

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks:...

High-level, abstract view of the value behavior of the decider function $F(p_1, p_2, p_3)$:



The value of each "$F_{i,j,k}$ region" can be **0 or 1** – from a purely combinatorial view.

Thus, the number of combinatorial possibilities is $2^8 = 256$. Only $2^6 = 64$ of these possibilities are viable for classification.

Viability: $F_{1,1,1}$ must be 0 and $F_{2,2,2}$ must be 1. $\Rightarrow 2^6 = 64$ viable combination results.

Since the '$\circ$' operator is commutative and associative, one can write the 8 combinations as follows:

$$L_1 \circ L_2 \circ L_3 = L$$
$$(L_1 \circ L_2) \circ H_3 \mathrel{\hat{=}} L \circ H = L \text{ or } H$$
$$(L_1 \circ L_3) \circ H_2 \mathrel{\hat{=}} L \circ H = L \text{ or } H$$
$$(L_2 \circ L_3) \circ H_1 \mathrel{\hat{=}} L \circ H = L \text{ or } H$$
$$L_1 \circ (H_2 \circ H_3) \mathrel{\hat{=}} L \circ H = L \text{ or } H$$
$$L_2 \circ (H_1 \circ H_3) \mathrel{\hat{=}} L \circ H = L \text{ or } H$$
$$L_3 \circ (H_1 \circ H_2) \mathrel{\hat{=}} L \circ H = L \text{ or } H$$
$$H_1 \circ H_2 \circ H_3 = H$$

It is instructive to briefly consider the three-scale / 3D case, where one must combine low and high values of three univariate decider functions, e.g., $F_1(p_1)$, $F_2(p_2)$ and $F_3(p_3)$:

$\circ$	$F =$
L_1, L_2, L_3	0
L_1, L_2, H_3	0 or 1
L_1, H_2, L_3	0 or 1
L_1, H_2, H_3	0 or 1
H_1, L_2, L_3	0 or 1
H_1, L_2, H_3	0 or 1
H_1, H_2, L_3	0 or 1
H_1, H_2, H_3	1

Again, the material classification context requires that the value of F must be **0 (1)** when all values to be combined are low (high). By induction, the following insight is gained:

When combining the low and high values of 2, 3, 4, 5,... univariate decider functions, the number of VIABLE F-values is $2^2, 2^6, 2^{10}, 2^{14}, \ldots$ The '$\circ$' is commutative and associative, and one can combine H-scale / H-dimensional data as

$$(L_{i_1} \circ L_{i_2} \circ \ldots \circ L_{i_k}) \circ (H_{i_{k+1}} \circ H_{i_{k+2}} \circ \ldots H_{i_H})$$
$$\mathrel{\hat{=}} L \circ H = L \text{ or } H.$$

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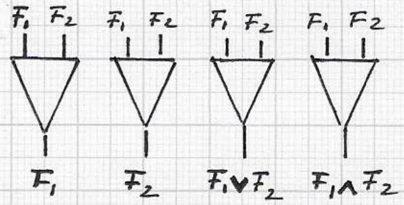
■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks: ...

Viewing F-values as output of logic/Boolean gates; considering the case of two functions F_1 and F_2:



The 4 viable results for the two-scale/2D case, see table on p. 2 from 6/18/2022.

Such gate-/circuit-based representations are advantageous when more complex decision-making and classification architectures must be illustrated effectively.

Generally, one should view '$\circ$' as a placeholder for the logical operators '$\vee$' and '$\wedge$'. Thus, these are all 2D cases:

$$L_1 \circ L_2 = 0$$

$$H_1 \circ H_2 = 1$$

$$L_1 \circ H_2 = 1, \text{ if '}\circ\text{' = OR}$$
$$= 0, \text{ if '}\circ\text{' = AND}$$

$$H_1 \circ L_2 = 1, \text{ if '}\circ\text{' = OR}$$
$$= 0, \text{ if '}\circ\text{' = AND}$$

These 4 viable possibilities are the ones shown in the illustration via gates.

• **Note.** One must keep in mind the "combinatorial explosion" that defines the computational complexity of the combination of the results generated by the univariate decider functions $F_i(p_i)$, $i = 1 \ldots H$. For example, when selecting only 4 of the H functions F_i, the number of viable combination results / combinations is $2^{10} = 1024$. Nevertheless, it can generally be assumed that a small number of the available H univariate decider functions F_i indeed suffices for classification; **a small number of scales should suffice to capture the unique characteristics of a material class.** The fact that eigenfunction-based data is available for H scales is **GOOD**: one can ultimately determine a small number of scales / scale data that ideally matches the demands of a classification task, driven by the number of classes and degree of class similarities.

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■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

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Six distinct cases exist in the three-scale/3D case for logically combining three values V_1, V_2 and V_3, see text (right, bottom). The actual values considered for V_1, V_2 and V_3 are once again low and high. The following table lists all possibilities:

V_1, V_2, V_3	①	②	③	④	⑤	⑥
L_1, L_2, L_3	L	L	L	L	L	L
L_1, L_2, H_3	H	L	L	H	L	L
L_1, H_2, L_3	H	L	L	L	L	L
L_1, H_2, H_3	H	H	H	H	L	L
H_1, L_2, L_3	H	L	H	L	L	L
H_1, L_2, H_3	H	H	H	H	H	L
H_1, H_2, L_3	H	L	H	H	H	L
H_1, H_2, H_3	H	H	H	H	H	H

The numbering of the cases – ①...⑥ – follows the numbering used in the text (right).

This table defines all possible ways to associate low or high values – L or H – with the 3D hyper-cube "regions" $F_{1,1,1} \dots F_{2,2,2}$; see p. 3 from 6/19/2022.

• **Note.** It is relevant to recall the relationship between the numerical/arithmetic operators '+' and '·' and the Boolean logic operators '∨' and '∧'. For the values 0 and 1, they can be summarized as:

+, ∨	0	1
0	0	1
1	1	1

·, ∧	0	1
0	0	0
1	0	1

The results of these operators are the same.

The 2×2 result matrices show that these two pairs of arithmetic-Boolean operators are "equivalent." Further, when breaking down a Boolean expression, given with the place-holder symbol '○', into the implied possible expressions with the actual '∨' and '∧' operators, ONE MUST KEEP IN MIND THE HIGHER PRECEDENCE OF THE '∧' OPERATOR.

Considering the three-scale/3D case, one combines values V_1, V_2 and V_3:

① $(V_1 \vee V_2) \vee V_3$; ① $V_1 \vee (V_2 \vee V_3)$;
② $(V_1 \vee V_2) \wedge V_3$; ③ $V_1 \vee (V_2 \wedge V_3)$;
④ $(V_1 \wedge V_2) \vee V_3$; ⑤ $V_1 \wedge (V_2 \vee V_3)$;
⑥ $(V_1 \wedge V_2) \wedge V_3$; ⑥ $V_1 \wedge (V_2 \wedge V_3)$.

...