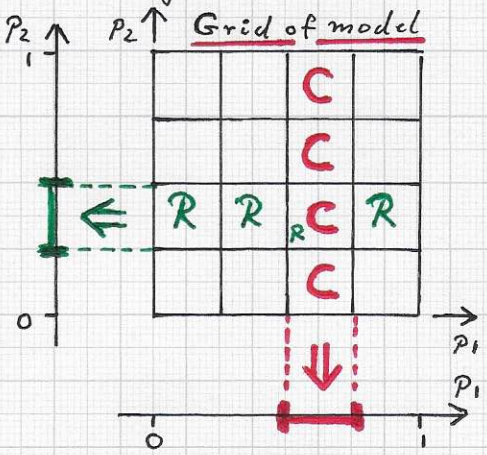
■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks: ...

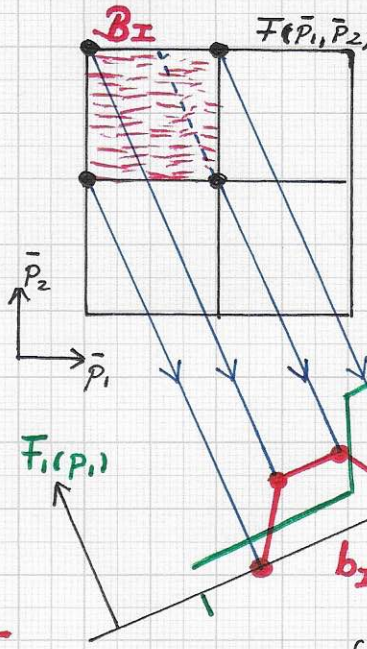
Geometrical explanation of the rank-deficient linear system / matrix encountered in the two simple synthetic examples discussed on the previous two pages:



Square grid cells in the same column C (row R) project to the same interval on the p_1-axis (p_2-axis), i.e., certain domains of box basis functions project to the same interval on one of the two axes. This fact leads to the observed rank deficiency of linear system matrices.

Thus, the projections of the box basis functions, i.e., their domains, on an axis should all be different from each other. The projected domains should not be identical – they should be "INTERLOCKING."

• **Note.** When reconstructing 2D/3D images from parallel or perspective CT projection data, a multitude of projection data is available for the (algebraic) reconstruction process. When employing a 2D grid of squares with associated box basis functions, for example, one can illustrate a reconstruction approach using best approximation principles as follows:



→ 2×2 reconstruction grid

→ box function B_I defined for square I

→ projection of B_I is b_I, a univariate function

→ F_1 is given

→ compute a best approximation of F_1 using basis $\{b_I\}$

→ $F_1^{app}(p_1) = \sum_I c_I b_I(p_1)$

→ once $\{c_I\}$ is known, $F(\bar{p}_1, \bar{p}_2)^{app}$ can be constructed as well

→ the normal equations define the solution:

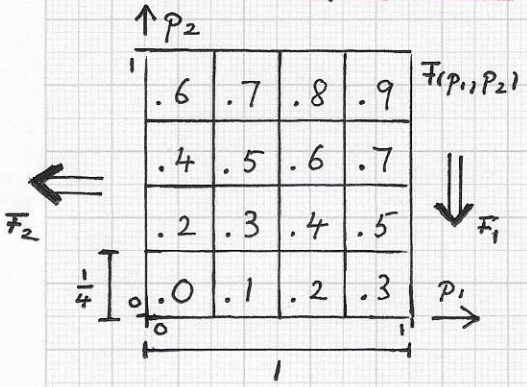
$$\begin{bmatrix} \langle b_1, b_1 \rangle & \cdots & \langle b_1, b_4 \rangle \\ \vdots & & \vdots \\ \langle b_4, b_1 \rangle & \cdots & \langle b_4, b_4 \rangle \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_4 \end{bmatrix} = \begin{bmatrix} \langle F_1, b_1 \rangle \\ \vdots \\ \langle F_1, b_4 \rangle \end{bmatrix}.$$

...

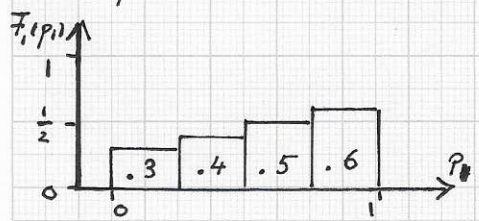
■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

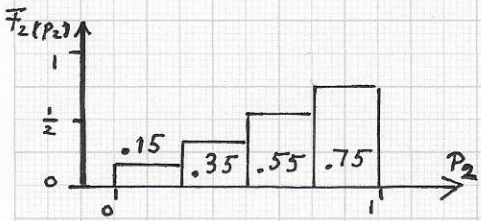
• Laplacian eigenfunctions and neural networks: ...

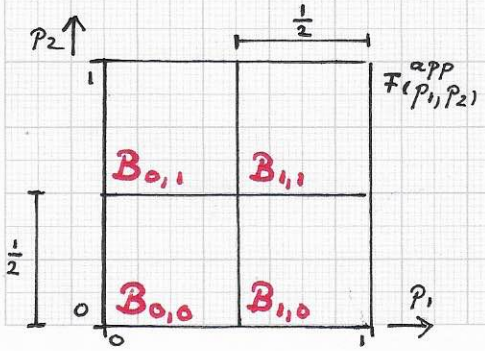
Another reconstruction example: 4x4 grid used to represent original function:



This function is **NOT** known. Given are its two projections $F_1(p_1)$ and $F_2(p_2)$ – each given in piecewise-constant form:







Goal: best approximation

$$F^{app} = \sum c_{I,J} B_{I,J}$$

Unfortunately, we must reconstruct, for example, the bivariate decider function $F(p_1,p_2)$ when only the two projections $F_1(p_1)$ and $F_2(p_2)$ are given – i.e., projections with projection directions that are the p_1- and p_2-axis directions of the "coordinate system" used to represent $F(p_1,p_2)$. "Oblique projection data" is not available. Another example is discussed to describe our case, see figure (left). The goal is the construction of a best approximation $F^{app}(p_1,p_2)$ for an unknown function $F(p_1,p_2)$ that is only "seen" via its two projections $F_1(p_1)$ and $F_2(p_2)$. The model used for F^{app} is a box basis function model for a 2x2 grid. Thus, the best approximation is

$$F^{app}(p_1,p_2) = \sum_{J=0}^{1} \sum_{I=0}^{1} c_{I,J} \cdot B_{I,J}(p_1,p_2),$$

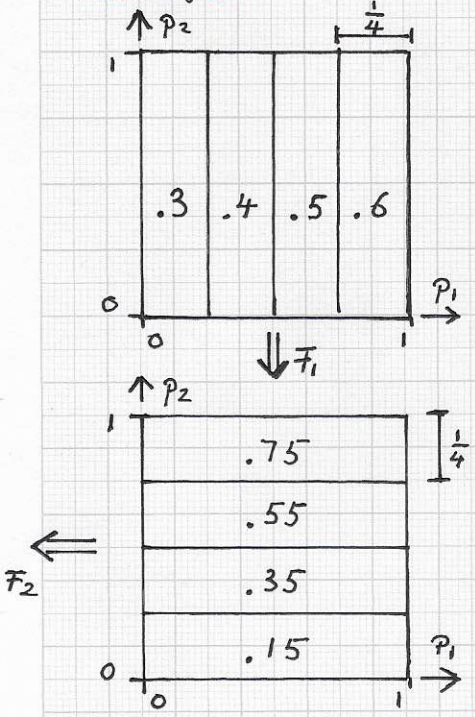
where $B_{I,J}$ has the value 1 in its associated domain square (and the value 0 outside its domain square).

...

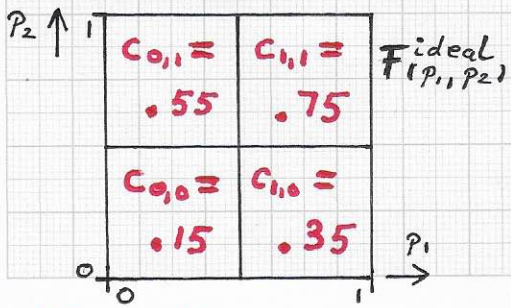
■ OBJECT AND MATERIAL EIGENFUNCTIONS - Cont'd.

• Laplacian eigenfunctions and neural networks:...

One can also visualize $F_1(p_1)$ and $F_2(p_2)$ – representing "column and row averages" as follows:



These average values (integral values) define the resulting equations for the unknown values of the coefficients in the box basis expansion of the best approximation.



"Ideal" reconstruction for a 2×2 grid

The visualizations of $F_1(p_1)$ and $F_2(p_2)$ shown in the figures (left) emphasize that these two functions represent "column and row averages" of the unknown function $F(p_1,p_2)$. Further, in this synthetic, ideal example the values of the two integrals $\int_0^1 F_1(p_1)\,dp_1$ and $\int_0^1 F_2(p_2)\,dp_2$ are identical (as it should be in a perfect scenario):

$$\int_{p_1=0}^{1} F_1(p_1)\,dp_1 = \frac{1}{4}(.3+.4+.5+.6) = .45$$

$$\int_{p_2=0}^{1} F_2(p_2)\,dp_2 = \frac{1}{4}(.15+.35+.55+.75) = .45$$

This value is viewed as the integral value of $F(p_1,p_2)$:

$$\int_{p_2=0}^{1}\int_{p_1=0}^{1} F(p_1,p_2)\,dp_1\,dp_2 = .45$$

The following equations result:

$$\frac{1}{4}\cdot(c_{0,0}+c_{1,0}+c_{0,1}+c_{1,1}) = .45$$

$$\frac{1}{4}\cdot(c_{0,0}+c_{1,0}) = \frac{1}{4}\cdot(.15+.35)$$

$$\frac{1}{4}\cdot(c_{0,1}+c_{1,1}) = \frac{1}{4}\cdot(.55+.75)$$

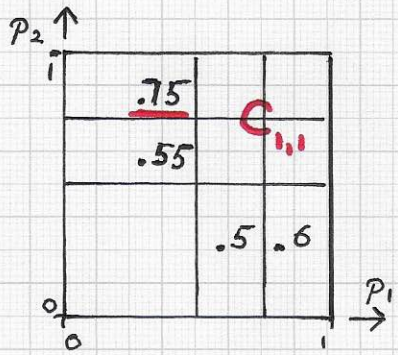
$$\frac{1}{4}\cdot(c_{0,0}+c_{0,1}) = \frac{1}{4}(.3+.4)$$

$$\frac{1}{4}\cdot(c_{1,0}+c_{1,1}) = \frac{1}{4}(.5+.6)$$

...

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks: ...



The linear system to be solved for the unknown values of $c_{I,J}$ leaves one value as a degree of freedom, e.g., the value of $c_{1,1}$.

By choosing the maximum of the shown four "strip values" – .75 – as value for $c_{1,1}$ one obtains as specific solution:

$$c_{1,1} = .75$$
$$c_{0,0} = .15$$
$$c_{1,0} = .35$$
$$c_{0,1} = .55$$

It is remarkable that this specific solution, a reconstruction based on the adhoc definition $c_{1,1} = .75$ equals the "ideal" reconstruction $F^{ideal}_{(p_1,p_2)}$. (See previous page.)

In matrix notation, this linear equation system is

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_{0,0} \\ c_{1,0} \\ c_{0,1} \\ c_{1,1} \end{bmatrix} = \begin{bmatrix} 1.8 \\ .5 \\ 1.3 \\ .7 \\ 1.1 \end{bmatrix}.$$

This system is under-determined. One can only obtain a parametric solution, with one of the four unknowns serving as "free" parameter:

$$c_{0,0} = c_{1,1} - .6 \; ; \; c_{1,0} = 1.1 - c_{1,1} \; ; \; c_{0,1} = 1.3 - c_{1,1}$$

Once the value of $c_{1,1}$ – the "free" parameter – is defined, the other three values are defined as well. The left figure (top) shows the coefficient $c_{1,1}$ in its associated domain square, together with the coefficient values of the four "strips" that intersect in and cover the top-right quadrant of the overall domain $[0,1]^2$. AN ADHOC CHOICE IS:

$$c_{1,1} = \max\{.5, .6, .55, .75\} = .75$$

Other choices for $c_{1,1}$ are possible, but the value .75 produces the "ideal" 2×2 reconstruction.

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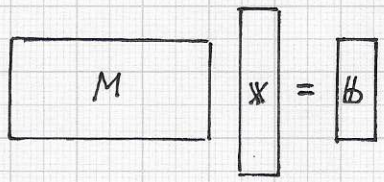
OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks:...

Solving underdetermined systems of linear equations via normal equations (Gauss):



Given underdetermined system $M\mathbf{x} = \mathbf{b}$

Define/select a vector $\bar{\mathbf{x}}$ that is an "IDEAL" to which a best possible vector $\mathbf{x}$ should be computed:

$\bar{\mathbf{x}} = \ldots$

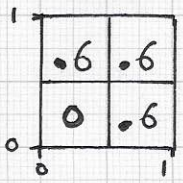
A priori knowledge can be used to define $\bar{\mathbf{x}}$.

• Step I:

Solve $MM^T\mathbf{y} = \mathbf{b} - M\bar{\mathbf{x}}$.

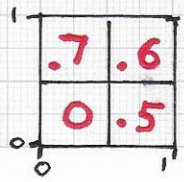
• Step II:

Compute $\mathbf{x} = \bar{\mathbf{x}} + M^T\mathbf{y}$.



$\bar{\mathbf{x}} = (0, .6, .6, .6)^T$ chosen as "IDEAL"

$\Rightarrow \int_0^1 \int_0^1 \sum_{j=0}^{1} \sum_{i=0}^{1} c_{i,j} B_{i,j} = .45$



$\mathbf{x} = (0, .5, .7, .6)^T$

= best approximation of $\bar{\mathbf{x}}$ considering the given equation system

Another method one can use to calculate a solution to such an underdetermined linear equation system is GAUSS' method that computes a "best solution" via the normal equations, see left figure. We use this method to tackle the problem discussed on the previous two pages. Equations 2, 3, 4 and 5 are:

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_{0,0} \\ c_{1,0} \\ c_{0,1} \\ c_{1,1} \end{bmatrix} = \begin{bmatrix} .5 \\ 1.3 \\ .7 \\ 1.1 \end{bmatrix}$$

When applying Gaussian elimination, one obtains

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} c_{0,0} \\ c_{1,0} \\ c_{0,1} \\ c_{1,1} \end{bmatrix} = \begin{bmatrix} .5 \\ 1.1 \\ 1.3 \end{bmatrix} \Leftrightarrow M\mathbf{x} = \mathbf{b}.$$

For example, one can define the value of an "IDEAL" as $\bar{\mathbf{x}} = (0, .6, .6, .6)^T$. The left figure (bottom) shows this choice – "qualitatively being an **OR**."

The resulting equation system for $\mathbf{y}$ is:

$$MM^T\mathbf{y} = \mathbf{b} - M\bar{\mathbf{x}}$$

$$\Leftrightarrow \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -.1 \\ -.1 \\ .1 \end{bmatrix} \Rightarrow \mathbf{y} = \begin{bmatrix} 0 \\ -.1 \\ .1 \end{bmatrix}.$$

$$\Rightarrow \mathbf{x} = \begin{bmatrix} 0 \\ .6 \\ .6 \\ .6 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -.1 \\ .1 \end{bmatrix} = \begin{bmatrix} 0 \\ .5 \\ .7 \\ .6 \end{bmatrix}.$$

...