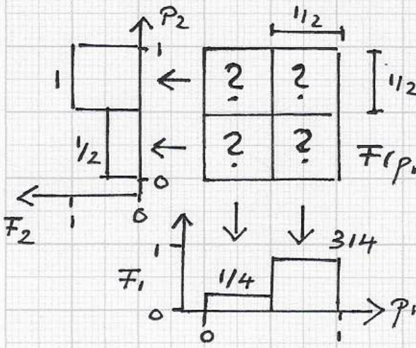
■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks:...

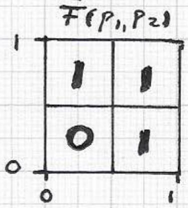
Given data that is **INCONSISTENT**:



The functions F_1 and F_2 represent integral properties in "column and row directions." To be consistent the sums of all column integrals and row integrals must be the same – since these sums are the unique integral of $F(p_1,p_2)$.

Here, F_1 and F_2 are NOT consistent.

Specified or known IDEAL function $\overline{F}(p_1,p_2)$:



The integral of this IDEAL function is NOT consistent with the functions F_1, F_2.

⇒ **Functions F_1, F_2 and $\overline{F}$ must be scaled so their integral properties are consistent.**

This GAUSS method for computing a best approximation of a good, meaningful solution of an underdetermined linear equation system requires us to enforce certain "consistency" conditions. More specifically, the given functions $F_1(p_1)$, $F_2(p_2)$, ... and the value of the IDEAL $\overline{x}$ must share specific integral properties to ensure that GAUSS' method is applicable. We must keep in mind that we have NO KNOWLEDGE of the exact nature of $F(p_1, p_2, \ldots)$, except via the univarite functions $F_i(p_i)$. We describe another simple bivariate example to clarify the issue of "consistency." The left figures show an example where a necessary consistency requirement is not satisfied: The three functions F_1 and F_2 (univariate decider functions) and $\overline{F}$ (postulated IDEAL decider function, an "OR" function) have different integral values.

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Bernd Hamann

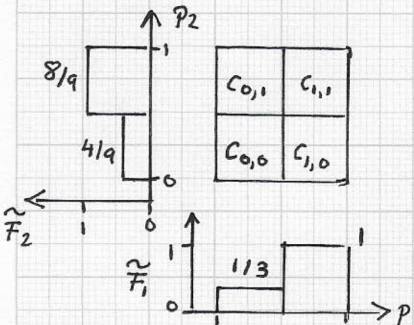
Stratovan

7/2/2022

OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks:...

Using the consistent, scaled functions $\tilde{F}_1$, $\tilde{F}_2$, $\tilde{\bar{F}}$ (see right), one obtains:



$c_{0,1}$	$c_{1,1}$
$c_{0,0}$	$c_{1,0}$

$$\Rightarrow \frac{1}{4}\cdot\begin{bmatrix}1&1&0&0\\0&0&1&1\\1&0&1&0\\0&1&0&1\end{bmatrix}\cdot\begin{bmatrix}c_{0,0}\\c_{1,0}\\c_{0,1}\\c_{1,1}\end{bmatrix}=\frac{1}{2}\cdot\begin{bmatrix}4/9\\8/9\\1/3\\1\end{bmatrix}$$

• Gaussian elimination leads to this system:

$$\begin{bmatrix}1&1&0&0\\0&1&0&1\\0&0&1&1\end{bmatrix}\cdot\begin{bmatrix}c_{0,0}\\c_{1,0}\\c_{0,1}\\c_{1,1}\end{bmatrix}=\begin{bmatrix}8/9\\2\\16/9\end{bmatrix}$$

$$M \quad \mathbb{y} \quad = \quad \mathbb{b}$$

• Solve $MM^T\mathbb{y} = \mathbb{b} - M\tilde{\tilde{\mathbb{x}}}$:

$$\begin{bmatrix}2&1&0\\1&2&1\\0&1&2\end{bmatrix}\mathbb{y}=\begin{bmatrix}0\\2/9\\0\end{bmatrix}$$

$$\Rightarrow \mathbb{y} = (-1/9,\ 2/9,\ -1/9)^T$$

• Solution $\tilde{\mathbb{x}} = (c_{0,0}, c_{1,0}, c_{0,1}, c_{1,1})^T$:

$$\tilde{\mathbb{x}} = \tilde{\tilde{\mathbb{x}}} + M^T\mathbb{y}$$

!

$$= \begin{bmatrix}0\\8/9\\8/9\\8/9\end{bmatrix} + \begin{bmatrix}1&0&0\\1&1&0\\0&0&1\\0&1&1\end{bmatrix}\begin{bmatrix}-1/9\\2/9\\-1/9\end{bmatrix}$$

$$= (-1/9,\ 1,\ 7/9,\ 1)^T$$

These are the relevant integral properties of the given functions:

• $\int F_1 = \frac{1}{2}\cdot(\frac{1}{4}+\frac{3}{4}) = \frac{1}{2}$

• $\int F_2 = \frac{1}{2}\cdot(\frac{1}{2}+1) = \frac{3}{4}$

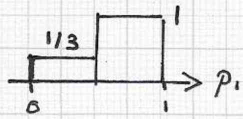
• $\int \bar{F} = \frac{1}{4}\cdot(0+1+1+1) = \frac{3}{4}$

$\phi = 2/3$

One "**can**" use the average value $\phi = 2/3$ as the CONSISTENT integral value of functions $\tilde{F}_1$, $\tilde{F}_2$ and $\tilde{\bar{F}}$ – properly scaled versions of the original functions F_1, F_2 and $\bar{F}$, respectively. Using s as needed scaling factor, one obtains:

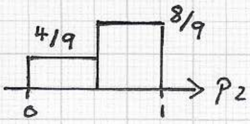
• $\int \tilde{F}_1 = \frac{2}{3} = s\cdot\frac{1}{2} \Rightarrow s = \frac{4}{3}$

$\Rightarrow \tilde{F}_1 = \frac{4}{3}F_1 =$



• $\int \tilde{F}_2 = \frac{2}{3} = s\cdot\frac{3}{4} \Rightarrow s = \frac{8}{9}$

$\Rightarrow \tilde{F}_2 = \frac{8}{9}F_2 =$



• $\int \tilde{\bar{F}} = \frac{2}{3} = s\cdot\frac{3}{4} \Rightarrow s = \frac{8}{9}$

$\Rightarrow \tilde{\bar{F}} = \frac{8}{9}\bar{F} =$

8/9	8/9
0	8/9

$\Rightarrow \tilde{\tilde{\mathbb{x}}} = \begin{bmatrix}0\\8/9\\8/9\\8/9\end{bmatrix}$

Having enforced consistency, it is now possible to perform GAUSS' method to solve the underdetermined linear equation system (left). ...

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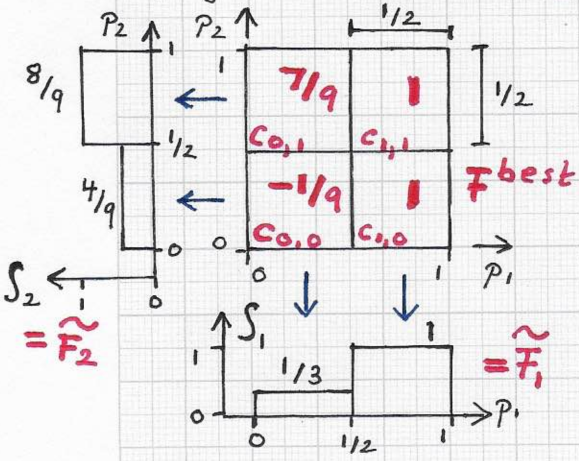
OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

- Laplacian eigenfunctions and neural networks: ...

Solution from previous page shown in 2D domain:



$$S_1 = \int_0^1 F^{best}(p_1,p_2)\,dp_2$$

$$S_2 = \int_0^1 F^{best}(p_1,p_2)\,dp_1$$

It is interesting that the calculated function $F^{best}(p_1,p_2)$ has integral properties – column and row integral properties – that reproduce the "input" univariate functions $\tilde{F}_1$ and $\tilde{F}_2$.

!

We used an average value – the average value $\phi = 2/3$ – to enforce "integral consistency" and thus the necessary COMPATIBILITY of the input to the best solution calculations. One can enforce compatibility in various ways – that could also ensure that all coefficients are non-negative.

One can perform the test to determine whether the calculated best solution **indeed** has the integral properties of $\tilde{F}_1$, $\tilde{F}_2$ and $\tilde{\tilde{F}}$. We call the best solution $F^{best}(p_1,p_2)$. The figure (left) illustrates the relevant functions and provides the numerical values needed for the test.

- First, the two discrete values of S_1 – $1/3$ and 1 – are identical to those of $\tilde{F}_1$; thus, $\tilde{F}_1$'s properties are all perfectly reproduced.
  $\left(\frac{1}{3} = \frac{1}{2}\left(-\frac{1}{9}+\frac{7}{9}\right),\ 1 = \frac{1}{2}(1+1)\right)$
- Second, the two discrete values of S_2 – $4/9$ and $8/9$ – are identical to those of $\tilde{F}_2$; thus, $\tilde{F}_2$'s properties are all perfectly reproduced.
  $\left(\frac{4}{9} = \frac{1}{2}\left(-\frac{1}{9}+1\right),\ \frac{8}{9} = \frac{1}{2}\left(\frac{7}{9}+1\right)\right)$
- Third, the integral value of $F^{best}(p_1,p_2)$ is $\frac{1}{4}\left(-\frac{1}{9}+1+\frac{7}{9}+1\right) = \frac{2}{3}$, which is equal to the integral value of $\tilde{\tilde{F}}$.

In summary, F^{best} passes the test.

The fact that F^{best} has a negative coefficient value ($c_{0,0} = -1/9$) is "undesirable" but can be "handled."

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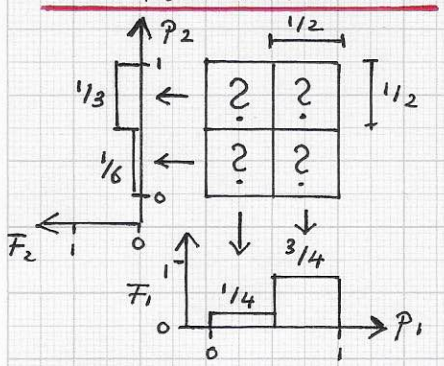
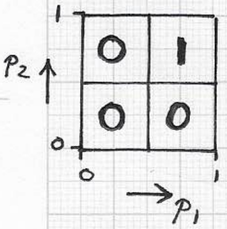
■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• **Laplacian eigenfunctions and neural networks: ...**

Given inconsistent data:

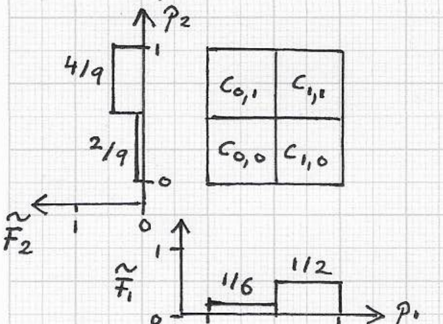


IDEAL $\overline{F}$



The functions F_1, F_2 and IDEAL are inconsistent with respect to their integral properties.

After the enforcement of compatibility one obtains the transformed set-up:



$$\Rightarrow \frac{1}{4}\cdot\begin{bmatrix}1&1&0&0\\0&0&1&1\\1&0&1&0\\0&1&0&1\end{bmatrix}\begin{bmatrix}c_{0,0}\\c_{1,0}\\c_{0,1}\\c_{1,1}\end{bmatrix}=\frac{1}{2}\cdot\begin{bmatrix}2/9\\4/9\\1/6\\1/2\end{bmatrix}$$

Gaussian elimination produces the expected under-determined linear system.

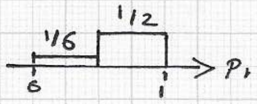
It is instructive to consider another numerical **2D** example, where the specified **IDEAL** decider function represents a logical **AND** function. The left figure shows the given data for this additional example. The inconsistent integral values are:

- $\int F_1 = \frac{1}{2}\cdot\left(\frac{1}{4}+\frac{3}{4}\right)=\frac{1}{2}$
- $\int F_2 = \frac{1}{2}\cdot\left(\frac{1}{6}+\frac{1}{3}\right)=\frac{1}{4}$
- $\int \overline{F} = \frac{1}{4}\cdot(0+0+0+1)=\frac{1}{4}$

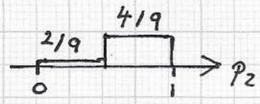
$\Big\}\ \Rightarrow \phi = \frac{1}{3}$

Again, to make all functions and data compatible, we use the average integral value $\phi=\frac{1}{3}$ and define the needed scaled functions of F_1, F_2 and $\overline{F}$:

- $\int \tilde{F}_1 = \frac{1}{3} = s\cdot\frac{1}{2} \Rightarrow s=\frac{2}{3}$

  $\Rightarrow \tilde{F}_1 = \frac{2}{3}F_1 =$ 

- $\int \tilde{F}_2 = \frac{1}{3} = s\cdot\frac{1}{4} \Rightarrow s=\frac{4}{3}$

  $\Rightarrow \tilde{F}_2 = \frac{4}{3}F_2 =$ 

- $\int \tilde{\overline{F}} = \frac{1}{3} = s\cdot\frac{1}{4} \Rightarrow s=\frac{4}{3}$

  $$\Rightarrow \tilde{\overline{F}} = \frac{4}{3}\overline{F} = \begin{array}{|c|c|}\hline 0 & \frac{4}{3}\\\hline 0 & 0\\\hline\end{array} \Rightarrow \tilde{\overline{x}} = \begin{bmatrix}0\\0\\0\\4/3\end{bmatrix}$$

...

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks:...

$$M\mathbf{y} = \mathbf{b}$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} c_{0,0} \\ c_{1,0} \\ c_{0,1} \\ c_{1,1} \end{bmatrix} = \begin{bmatrix} 4/9 \\ 1 \\ 8/9 \end{bmatrix}$$

Resulting underdetermined system

• Solve $MM^T\mathbf{y} = \mathbf{b} - M\tilde{\tilde{\mathbf{x}}}$:

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \mathbf{y} = \begin{bmatrix} 4/9 \\ -1/3 \\ -4/9 \end{bmatrix}$$

$$\Rightarrow \mathbf{y} = (7/18, -1/3, -1/18)^T$$

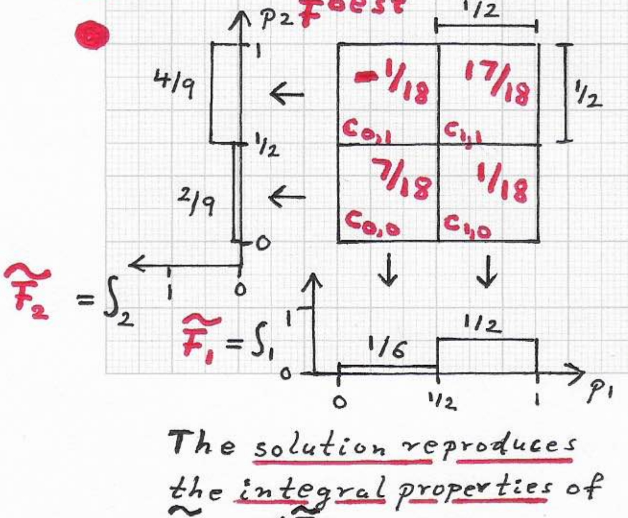
• Compute solution for the transformed, i.e., scaled, functions:

$$\tilde{\mathbf{x}} = \tilde{\tilde{\mathbf{x}}} + M^T\mathbf{y} =$$

$$= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 4/3 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 7/18 \\ -1/3 \\ -1/18 \end{bmatrix}$$

$$= (7/18, 1/18, -1/18, 17/18)^T$$

• Solution in 2D domain:



The solution reproduces the integral properties of $\tilde{F}_1$ and $\tilde{F}_2$.

Setting up the four equations for the coefficients $c_{0,0}$, $c_{1,0}$, $c_{0,1}$ and $c_{1,1}$ and performing Gaussian eliminations leads to the system shown in the figure (left). The figure summarizes the steps of Gauss' solution approach. The generated best solution $F^{best}(p_1,p_2)$ once again reproduces the univariate functions $\tilde{F}_1$ and $\tilde{F}_2$ exactly – and therefore also all column and row integral values defining $\tilde{F}_1$ and $\tilde{F}_2$ (see left figure, bottom). Further, the integral value of F^{best} is $\int F^{best} = \frac{1}{4}(7/18 + 1/18 - 1/18 + 17/18) = 1/3 = \int \tilde{\tilde{F}}$. The value of $c_{0,1}$ is negative, and this fact must be handled in a data analysis/classification application of F^{best}. **In this example, the calculated function F^{best} does NOT represent an AND function: the values of $c_{1,0}$, $c_{0,1}$ and $c_{1,1}$ are very close to an AND representation, but $c_{0,0}$'s value is not close to 0. Of course, the functions $\tilde{F}_1$ and $\tilde{F}_2$ imply $c_{0,0}$'s value.**

...