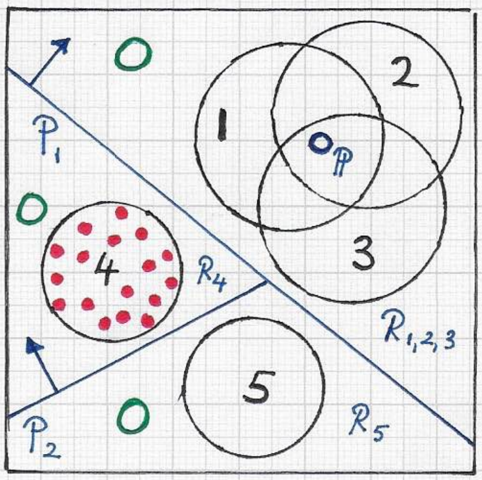
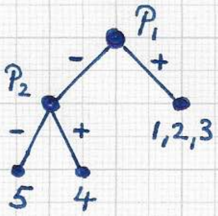
■ OBJECT AND MATERIAL EIGENFUNCTIONS — Cont'd.

• Laplacian eigenfunctions and neural networks:...

Simple 2D BSP and its use in data classification:





The bounding hyper-spheres of the stored sample histogram points for classes 1, 2 and 3 intersect.

For example, one could use a multi-index for the tiles of the BSP, indicating, for some tiles, a set of class hyper-spheres that mutually intersect with each other — like tile $R_{1,2,3}$ in this figure.

The unclassified point P is quickly identified as a point in tile $R_{1,2,3}$ — but it fails all the test that would designate it as a class-0 point. (Point P is not outside the minimal bounding box of the three intersecting hyper-spheres; in fact, P is inside all three hyper-spheres.)

• Note. Since the binary tree representation of a BSP makes it possible to determine the tile in the BSP containing an unclassified histogram point in logarithmic search time, the efficiency gain over linear search increases with an increasing number of classes/tiles. For example, when considering $64 = 2^6$ classes, the tile of a specific class and the given unclassified histogram point can be determined in 6 (comparison) steps, on average, using a BSP tree. Further, in case an optimally balanced BSP tree is desired for optimal search time behavior, such a balanced tree can be precomputed for the material class samples data stored in the database. The example shown in the left figure demonstrates the case of multiple disks/hyper-spheres lying in one tile — as a consequence of mutually overlapping non-separable disks.

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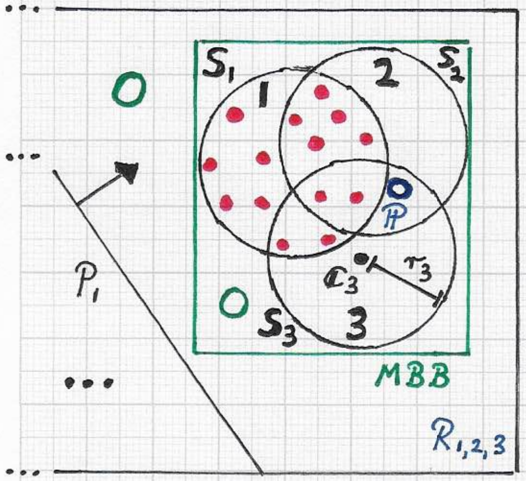
OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

- Laplacian eigenfunctions and neural networks:...

Illustration of tests applied to point $\mathbb{p}$ for its classification:



Point $\mathbb{p}$ is inside S_2 and S_3, and one must perform all steps that are necessary to calculate probability values for $\mathbb{p}$ determining the likelihood of belonging to class 2 or 3.

- **Note.** Coefficient value histogram data are known and stored for H scales – for all samples and for all material classes. This figure captures a possible scenario merely for one specific scale. For example, for a different scale the bounding hyper-spheres for class-1, class-2 and class-3 histogram point set might not intersect at all, and the unclassified point $\mathbb{p}$ might reside in a class-0 region.

The left figure shows the upper-right part of the figure from the previous page in more detail. These facts characterize the sketched scenario concerned with the classification of the unclassified histogram point $\mathbb{p}$:

→ The BSP tree is used to determine the tile containing $\mathbb{p}$; the resulting tile is $R_{1,2,3}$.

→ Point $\mathbb{p}$ fails the first class-0 test, since the point does not lie outside the minimal (hyper-) bounding box (MBB).

→ Point $\mathbb{p}$ also fails the second class-0 test inside the MBB, since the point does not lie outside all (three) hyper-spheres that reside inside the MBB. The point $\mathbb{p}$ is

inside hyper-sphere S_i

$$\Longleftrightarrow \|\mathbb{p} - \mathbb{c}_i\|^2 < r_i^2 .$$

Here, $\mathbb{c}_i$ is the center point of hyper-sphere S_i. Point $\mathbb{p}$ is **inside S_2 and S_3**. (More precisely, one should say "ball" – not "sphere.")

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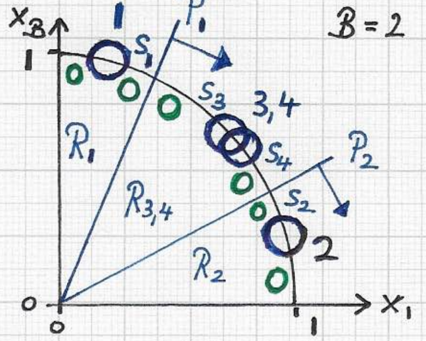
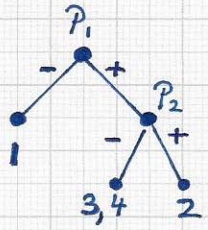
OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks:...

Abstract illustration of histogram points, with unit, normalized positional vectors, lying on a unit hyper-sphere in the embedding space:





It is possible to ignore the fact that all histogram points have unit positional vectors.

All sample points and an unclassified point lie on the unit hyper-sphere defined by $x_1^2 + \ldots + x_B^2 = 1$, $x_1, \ldots, x_B \geq 0$. Nevertheless, a BSP and an associated binary tree can be constructed in the B-dimensional embedding space, as shown.

Here, sample histogram points lie inside the bounding hyper-spheres $S_1, \ldots, S_4$. Thus, five classes are considered, i.e., classes $0, 1, \ldots, 4$.

Of course, as discussed before, certain necessary distance computations can use angles (cos values) of positional vectors involved.

→ Since point $\mathbb{P}$ only lies inside the region bounded by the hyper-spheres S_2 and S_3, one must perform a detailed classification procedure only for classes 2 and 3, for the specific scale illustrated in the figure on the previous page. Two probability values result from this procedure, i.e., probability values concerning likelihood of belonging to class 2 or class 3.

→ Once all available (or only some) scales have been considered for the classification of point $\mathbb{P}$, one effectively has calculated a probability table/matrix for $\mathbb{P}$:

class \ scale	1	2	...	H-1	H
0	p_0^1		...		p_0^H
1	p_1^1		...		p_1^H
⋮	⋮				⋮
C	p_C^1		...		p_C^H

Probabilities

A final classification of $\mathbb{P}$ is made by considering $H \cdot (C+1)$ values.

→ One can now construct "optimized" class probability functions $P_{cl} = P_{cl}(p_{cl}^1, \ldots, p_{cl}^H)$, $cl = 0 \ldots C$, defining $\mathbb{P}$'s combined probabilities for all classes.

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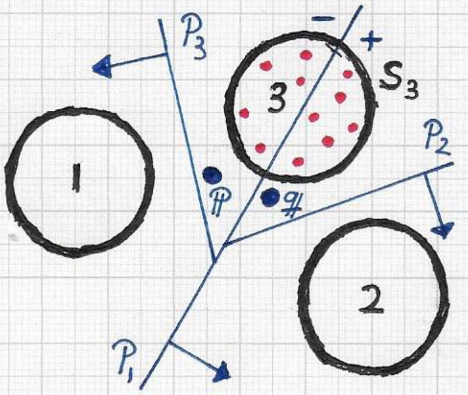
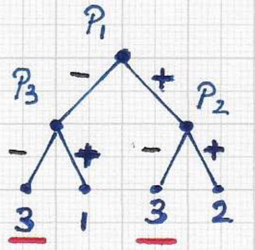
■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks:...

Possible singular, degenerate situation that must be handled:





Configuration where a separating hyperplane has sample points of class 3 "on both sides."

One possibility to deal with this case is to perform a "quasi-split" of the class-3 sample histogram points:

➜ Insert hyper-sphere S_3: S_1 lies partially in '−' and partially in '+' half-space of P_1.

➜ Insert S_3 "twice" by using two separating hyper-planes P_2 and P_3.

➜ The resulting updated BSP tree has TWO Leaves referencing the same class, i.e., class 3.

• **Note.** The paper "Discrete Sibson Interpolation" authored by Sung W. Park et al. (IEEE Transactions on Visualization and Computer Graphics 12 (2), pp. 243-253, 2006) discusses a closely related topic: the efficient construction of a discrete approximation of the Voronoi diagram for data interpolation. This method also uses a tree-based representation of space and scales to high-dimensional domains. The case shown in the left figure must be dealt with as a special case.

Since SAMPLE SETS are provided to represent the classes – and not merely a single sample point per class – a separating hyper-plane can have some sample points of a class in its '−' half-space and some sample points (of the same class) in its '+' half-space. In the figure, the sample points of class 3 are divided by hyper-plane P_1.

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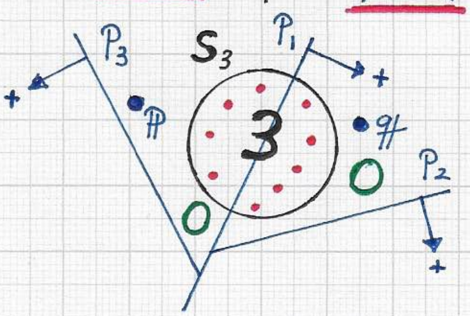
■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks:...

Using the BSP and tree to search for points p and q:

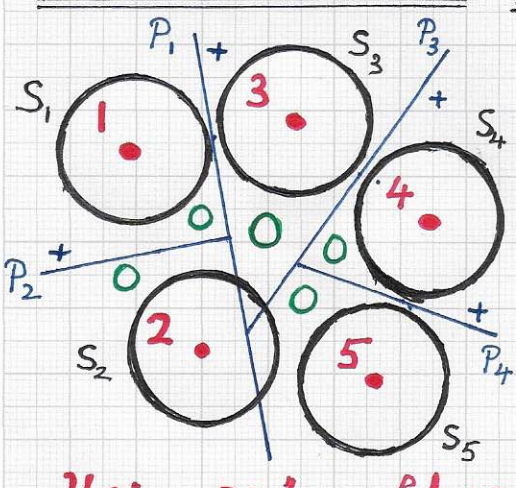


→ Search p in tree:
- '–' half-space of P_1
- '–' half-space of P_3
- done: S_3, class 3

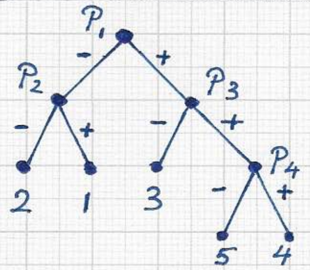
→ Search q in tree:
- '+' half-space of P_1
- '–' half-space of P_2
- done: S_3, class 3

Class 3, S_3, is determined for both p and q, since TWO leaves in the BSP tree refer to class 3 (samples).

The figure on the previous page includes the unclassified histogram points p and q. The shown BSP tree for these two points correctly determines class 3 as the class to be considered as the potential class that p and q might belong to: The tree refers to class 3 when identifying the hyper-sphere that must be used for these two points' classification. Unfortunately, with increasing numbers of sample histogram points, classes and radii of hyper-spheres etc., this issue must be addressed more frequently. An alternative approach for BSP definition is shown in the left figure. This approach uses the center points of the hyper-spheres for the definition. Regardless of the chosen approach, numerous special cases must be handled. ...



Using centers of hyper-spheres for BSP definition



BSP and tree obtained when splitting space in sequence P_1, P_2, P_3, P_4. Only the center of a hyper-sphere S_i is used for the construction of the BSP.