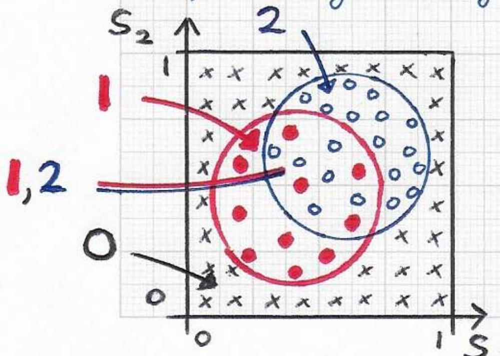
■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks:... Examples: Sampling. The last issue mentioned at the bottom of the previous page concerns the generation of a LARGE AND DENSE, STILL DISCRETE AND FINITE, set of $(S_1, \ldots, S_C)$ data by using classified segments belonging to classes $1, \ldots, C$ — reflecting the REAL-WORLD occurrence percentages of these classes $1, \ldots, C$. The resulting distribution of this discrete data set in $(S_1, \ldots, S_C)$-space can be used to define "densities" for the observed $(S_1, \ldots, S_C)$-tuples for a C-dimensional domain space. As a consequence, these "densities" can be used to define class probabilities: "GIVEN AN UNCLASSIFIED SEGMENT AND ITS COMPUTED $(S_1, \ldots, S_C)$-tuple, WHAT ARE THE SEGMENT'S $(C+1)$ CLASS PROBABILITIES $p_0, p_1, \ldots, p_C$?" This question can be answered via the distribution of the $(S_1, \ldots, S_C)$-tuples of the additional LARGE classified data set. We describe this approach by considering several scenarios of increasing complexity and generality. The left figure shows a simple example for three classes (0, 1, 2) and a 2D S-space (S_1, S_2). The S-tuples resulting from sampling lead to FOUR regions: $R_0, R_1, R_2, R_{1,2}$. THE REGION BOUNDARIES (circular arcs) DO NOT EXIST; they are imposed for context and probability computations...



Material classes 1 and 2 defining four regions.

OBJECT AND MATERIAL EIGENFUNCTIONS - Cont'd.

• Laplacian eigenfunctions and neural networks:... The four distinct regions shown in this example (last page) "tessellate" the $[0,1]^2$ domain into four unique areas of probability tuple (p_0, p_1, p_2) behavior. The four region-specific probability tuples are:

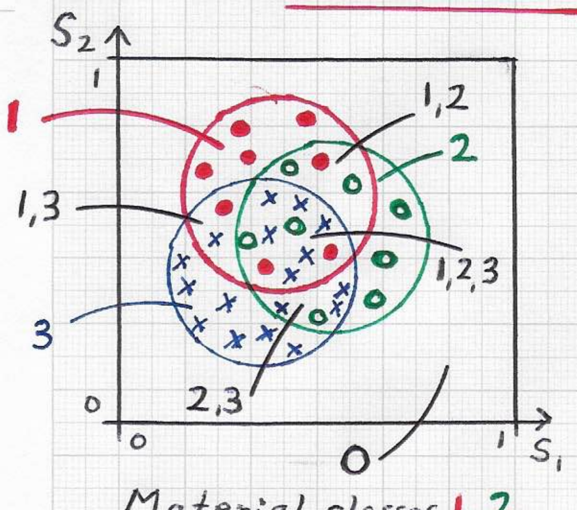
$R_0 \hat{=} (1,0,0)$
$R_1 \hat{=} (0,1,0)$
$R_2 \hat{=} (0,0,1)$
$R_{1,2} \hat{=} (0,.4,.6)$

Concerning region $R_{1,2}$, this "tile" contains 4 class-1 and 6 class-2 (s_1, s_2)-tuples. Thus, $p_1 = 4/10$ and $p_2 = 6/10$.
FOR EXAMPLE, IF AN UNCLASSIFIED SEGMENT HAS AN ASSOCIATED COMPUTED $-TUPLE THAT LIES IN REGION $R_{1,2}$, THEN THE PROBABILITIES OF THE SEGMENT FOR BELONGING TO CLASS 1 OR CLASS 2 WILL BE .4 AND .6, RESPECTIVELY.

• **Note.** In this "ideal" scenario, it is implicitly assumed that the density of $-tuples, representing a certain class in the $-domain, is CONSTANT. In other words, the number of class-1 $-tuples per unit area in the shown class-1 disk-region is CONSTANT. Generally, such $-tuple densities VARY in their associated $-domain regions. This fact must be considered in generalizations.

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks:... Further, the "boundaries" of such $\mathcal{S}$-domain regions containing all $\mathcal{S}$-tuples of the LARGE discrete classified dataset are not known, do not physically exist, and should not have to be estimated. **THUS**, WHEN AN UNCLASSIFIED SEGMENT IS GIVEN IN THE FORM OF ITS $\mathcal{S}$-TUPLE IN THE $\mathcal{S}$-DOMAIN, ONE MUST USE A PROPER LOCAL **NEIGHBORHOOD** IN THE $\mathcal{S}$-DOMAIN AND **COUNT** THE NUMBERS OF CLASS-1... CLASS-C $\mathcal{S}$-TUPLES IN THIS LOCAL NEIGHBORHOOD.



Material classes 1, 2 and 3 defining eight regions.

The left figure shows a scenario that involves classes 0, 1, 2 and 3. Again, circular-arc "boundaries" are merely shown for context. There are eight distinct regions defined by the "disk intersections" / the "overlap of the $\mathcal{S}$-tuple data." It is assumed, for simplicity, that class-tuple density is constant (per class). The 4-tuple probabilities are:

$R_0 \hat{=} (1,0,0,0)$,

$R_1 \hat{=} (0,1,0,0)$, $R_2 \hat{=} (0,0,1,0)$, $R_3 \hat{=} (0,0,0,1)$,

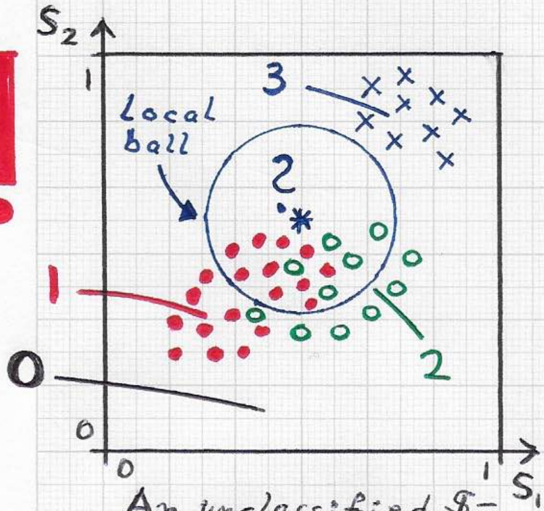
$R_{1,2} \hat{=} (0,1/3,2/3,0)$, $R_{1,3} \hat{=} (0,1/2,0,1/2)$, $R_{2,3} \hat{=} (0,0,1/4,3/4)$,

$R_{1,2,3} \hat{=} (0,1/5,1/5,3/5)$.

...

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks:... • SINCE "REGION BOUNDARIES" DO NOT EXIST AND $-TUPLE DENSITIES VARY FOR ALL CLASSES 1...C IN THE $-DOMAIN, ONE MUST — IN THE ABSENCE OF CONTINUOUSLY DEFINED DENSITY FUNCTIONS — USE A LOCAL SUBSET OF THE DISCRETE $-TUPLE DATA TO ESTABLISH A PROBABILITY TUPLE $(p_0, p_1, \dots, p_C)$ FOR THE $-TUPLE OF AN UNCLASSIFIED SEGMENT. The scenario illustrated in the left figure concerns four classes 0, 1, 2 and 3. One must determine a (p_0, p_1, p_2, p_3)-tuple for the (S_1, S_2)-tuple associated with *, representing the unclassified material. A local ball of specified radius is placed into the $-domain, having * as its center. The ball is used to estimate densities for classes 1, 2 and 3 in the interior of the ball — by counting how many class-1, class-2 and class-3 $-tuples lie inside the ball. The resulting number ratios define the p_{ci}-values. Here, one obtains $p_0 = 0$, $p_1 = {}^{10}/_{15}$, $p_2 = {}^{5}/_{15}$, $p_3 = 0$.



An unclassified $-tuple * is the center of a local ball used to calculate a probability tuple for *.

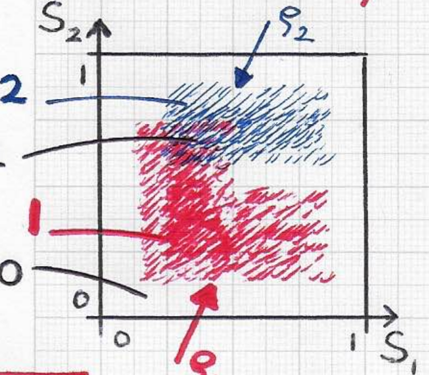
...

OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks: ...

• **This local ball-based method for estimating $\mathbb{S}$-tuple densities is still a "discrete approach" for calculating local pel-value estimates by counting $\mathbb{S}$-tuples inside the ball; the method is not computing actual density functions for a continuous $\mathbb{S}$-domain. In some application settings, it could be advantageous to pre-compute, store and use directly density functions for the entire $\mathbb{S}$-domain, i.e., $[0,1]^c$. The figure (left) provides a simple sketch of such density functions defined over potentially intersecting continuous domains in the overall $\mathbb{S}$-domain. Here, two density functions $\rho_1(S_1, S_2)$ and $\rho_2(S_1, S_2)$ define the $\mathbb{S}$-tuple densities for class-1 $\mathbb{S}$-tuples and class-2 $\mathbb{S}$-tuples, respectively. The figure includes four regions; the probabilities are:**

$$R_0 \triangleq (1,0,0), \quad R_1 \triangleq (0,1,0),$$
$$R_2 \triangleq (0,0,1), \quad R_{1,2} \triangleq \left(0, \rho_1/(\rho_1+\rho_2), \rho_2/(\rho_1+\rho_2)\right).$$



Use of density functions ρ_1 and ρ_2 for probability tuple computation. When domains of density functions intersect, one must use the ratios of their values for pel.

!

"ρ = no. of $\mathbb{S}$-tuples per unit area $\Rightarrow$ ρ is NOT normalized."

!

• **Note.** In computational cosmology, one often must compute density functions from discrete particle data. A good reference is: "Self-adaptive density estimation of particle data" by Peterka, Croubois, Li, Rangel and Cappello, SIAM J. of Scientific Computing, Vol. 38(5), 2016. We can employ such methods from cosmology.

...