OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks:... •Note. The "C-ball" example shown in the top figure of the previous page uses 2D "C-cubes" (squares) to explain the idea of extrapolating to a limit volume of zero. Of course, one must use real C-dimensional balls in an algorithm. The volumes of "C-balls" are given by the following formulas:

$$V_{2k} = \frac{\pi^k}{k!} r^{2k}, \; k=0,1,2,\ldots \Rightarrow V_0 = 1, V_2 = \pi r^2, V_4 = \frac{\pi^2}{2} r^4, \ldots$$

$$V_{2k+1} = \frac{2k!(4\pi)^k}{(2k+1)!} r^{2k+1}, \; k=0,1,2,\ldots \Rightarrow V_1 = 2r, V_3 = \frac{4\pi}{3} r^3, V_5 = \frac{8\pi^2}{15} r^5, \ldots$$

These volumes, using some fixed radius r, initially monotonically increase with increasing C-value, reach a maximum value, and then monotonically decrease with increasing C-value. Nevertheless, volumes V_C are always proportional to r^C. This behavior is interesting.

• Richardson and ITERATED Richardson extrapolation are allowing one, for example, to define and efficiently compute sequences of values that converge to a needed limit value for the derivative of a function $f(x)$ at a location $x = a$ or for the integral of $f(x)$ for the interval $a \le x \le b$. The fundamental idea of Richardson extrapolation is this: Devise a numerical method for the approximation of the quantity needed; this method uses a specific STEPSIZE h.

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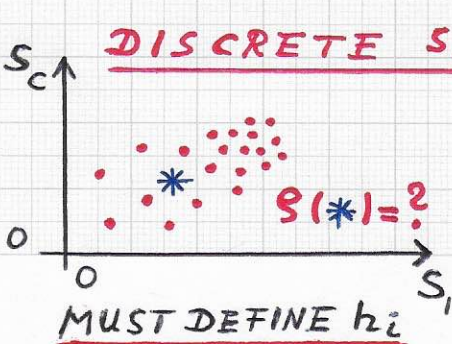
■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks:... The "trick" of Richardson extrapolation is the definition of a sequence of stepsizes $h_0, h_1, h_2, \ldots$ leading to corresponding approximations $A_0, A_1, A_2, \ldots$ of the quantity of interest. In general, stepsizes shrink in a geometrical fashion, i.e., $h_{i+1} = \alpha h_i$, $0 < \alpha < 1$. The extrapolated value "$A_\infty = A(h_\infty = 0)$" is the desired limit value – subject to certain conditions being satisfied by the function to be approximated and certain design principles being followed by the extrapolation method. FOR OUR PURPOSES, THE ESTIMATION OF DENSITY ESTIMATES $\varrho(*)$ FOR $\mathcal{S}$-TUPLES $* = (s_1, \ldots, s_c)$ IN THE $\mathcal{S}$-DOMAIN, ONE MUST ADJUST RICHARDSON EXTRAPOLATION IN SUCH A WAY THAT A CALCULATED SEQUENCE OF A_i APPROXIMATIONS CONVERGES TO $\varrho(*)$. THE CHALLENGE IN OUR CASE IS THE NECESSITY TO PROPERLY DEFINE "STEPSIZE" FOR OUR DISCRETE SAMPLE DATA IN THE $\mathcal{S}$-DOMAIN.



MUST DEFINE h_i

The left figure sketches our discrete sample setting: We must define a stepsize sequence $h_0, h_1, h_2, \ldots$ and resulting $\varrho(*)$ approximations $A_0, A_1, A_2, \ldots$ STOCHASTICALLY.

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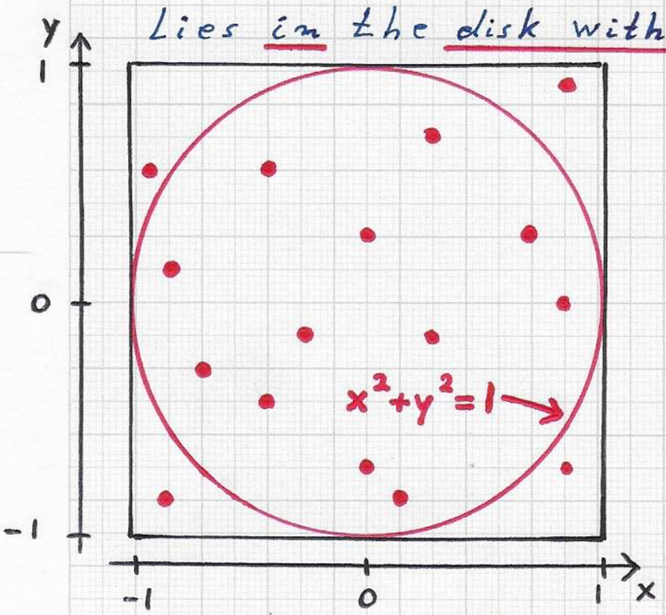
Stratovan

OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks:...

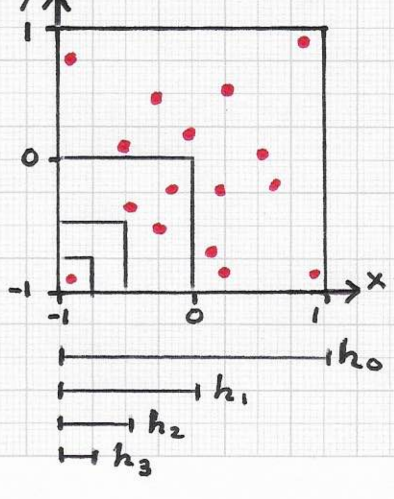
• **Note.** We introduce STOCHASTIC RICHARDSON EXTRAPOLATION via an example concerning the approximation of the value of π. We randomly generate points $(x,y)^T$, $-1 \leq x \leq 1$ and $-1 \leq y \leq 1$, and determine whether a random point $(x,y)^T$ satisfies $x^2+y^2 \leq 1$ — indicating that it lies in the disk with radius 1 and area π. The figure (left) explains the idea for approximating π's value: The total number of randomly generated points in the square $[-1,1]^2$ is n_S; the number of points satisfying $x^2+y^2 \leq 1$ is n_D. In the limit, the following equation will hold:



$\pi \approx 4 n_D / n_S = 4\,{}^{12}/_{16} = 3$

$$\pi/4 = n_D/n_S \Rightarrow \pi = 4\, n_D/n_S.$$

The next figure (left) illustrates a possibility to define a stepsize sequence together with an associated number sequence of n_S-values. Here, $h_0 = 2$, $h_1 = 1$, $h_2 = 1/2$, $h_3 = 1/4, \ldots, h_i = h_{i-1}/2$; $n_S^0 = 1$, $n_S^1 = 4$, $n_S^2 = 16$, $n_S^3 = 64, \ldots, n_S^i = 2^{2i}$. In summary, stepsize and n_S-values are:

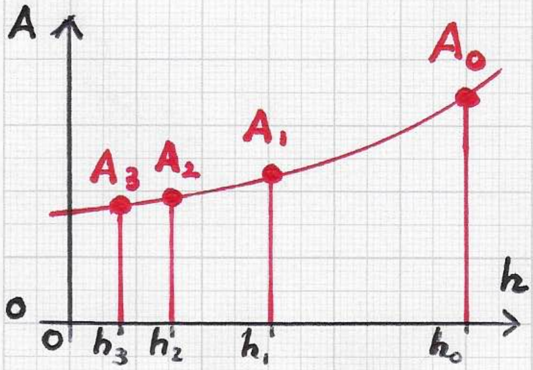


$$h_i = 2^{1-i}, \quad n_S^i = 2^{2i}, \quad i = 0, 1, 2, \ldots$$

...

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks:...

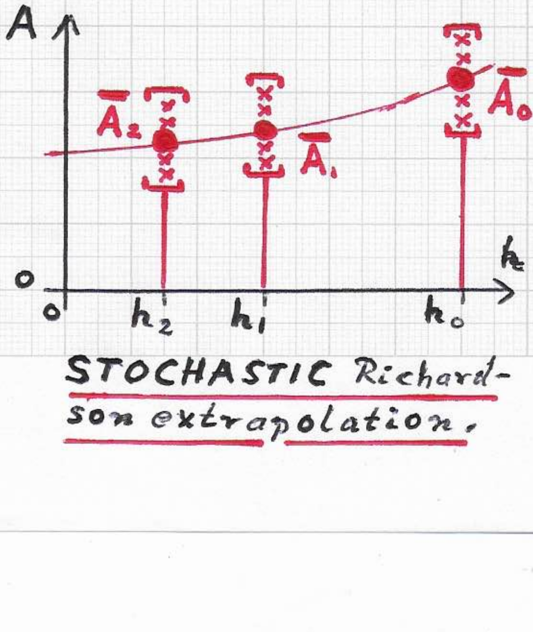


One RANDOM sequence of A_i-values

Thus, for a sequence h_0, h_1, h_2,... one obtains the sequence of corresponding π approximations:

$$A_0 = 4\, n_D^0 / n_S^0 ,\ldots, A_i = 4\, n_D^i / n_S^i ,\ldots$$

The sequence $A_0, A_1, A_2, A_3, \ldots$ is RANDOMLY GENERATED, since the coordinates of the n_S^i points are randomly generated. The figure (top) shows such a sequence. We can produce a multitude of such sequences by performing the random coordinate value generation of n_S^i points many times. This repeated random point set generation approach leads to a set of A_0-values, a set of A_1-values, a set of A_3-values etc. Richardson extrapolation can thus be generalized to STOCHASTIC Richardson extrapolation by using an AVERAGE of multiple A_i-values. Specifically, one generates a multitude of A_0-values and calculates their average $\overline{A}_0$. Eventually, one obtains the sequence $\overline{A}_0, \overline{A}_1, \overline{A}_2, \ldots$ and performs Richardson extrapolation with these $\overline{A}_i$-values, see figure.



STOCHASTIC Richardson extrapolation.

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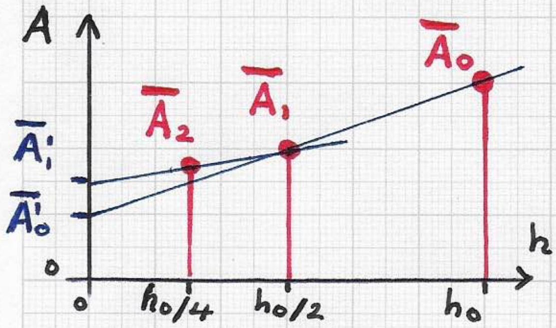
■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks:...



Richardson extrapolation where (i) $h_i = h_0/2^i$ and (ii) $\bar{A}_i$ and $\bar{A}_{i+1}$ are linearly interpolated.

The figure (left) sketches a special Richardson extrapolation scheme, based on step-sizes h_0, $h_0/2$, $h_0/4$, ... and linear extrapolation of two values $\bar{A}_i$ and $\bar{A}_{i+1}$ for $h=0$. (The blue lines in the figure represent this linear extrapolation.) The values $\bar{A}_i^0 = \bar{A}_i$ and $\bar{A}_{i+1}^0 = \bar{A}_{i+1}$ are linearly interpolated; and the value of the linear interpolation function at $h=0$ is $\bar{A}_i^1$. This extrapolation procedure is subsequently applied again to the $\bar{A}_0^1$-, $\bar{A}_1^1$-, $\bar{A}_2^1$-, ...-values, defining the ITERATED Richardson extrapolation method. For the special setting illustrated in the figure, one obtains the formula for $\bar{A}_i^j$:

$i \backslash j$	0	1	2	3
0	$\bar{A}_0^0$	$\bar{A}_0^1$	$\bar{A}_0^2$	$\bar{A}_0^3$
1	$\bar{A}_1^0$	$\bar{A}_1^1$	$\bar{A}_1^2$	
2	$\bar{A}_2^0$	$\bar{A}_2^1$		
3	$\bar{A}_3^0$			

...

Iterated stochastic Richardson extrapolation.

$$\bar{A}_i^j = \frac{\bar{A}_{i+1}^{j-1} - \bar{A}_i^{j-1}/2^j}{1 - 1/2^j}.$$

Computationally, one does the following: i) compute $\bar{A}_i^0$, $i = 0 \ldots N$;

ii) for $j = 1 \ldots N$
  for $i = 0 \ldots (N-j)$
    compute $\bar{A}_i^j$;

iii) result: $\bar{A}_0^N$.

The scheme must be adapted to the estimation of density values $\varrho(s_1, \ldots, s_c)$.

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