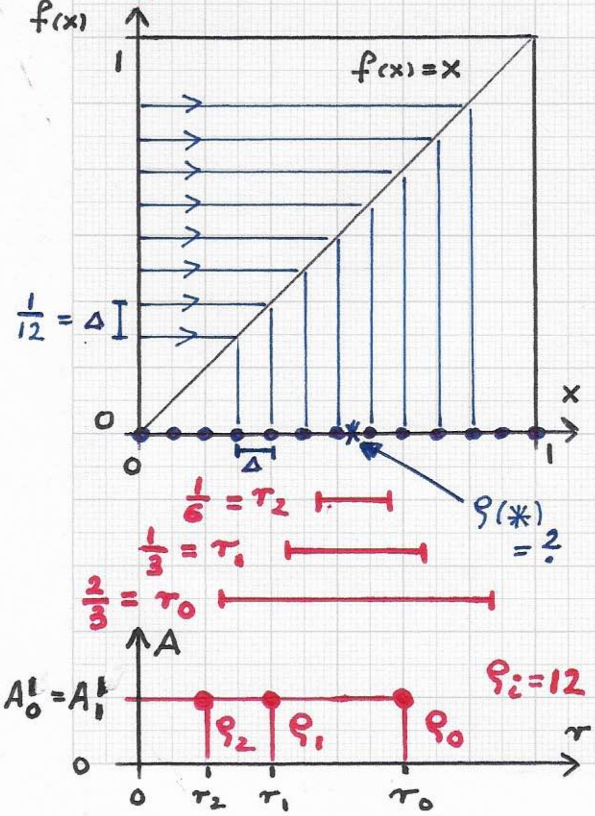
■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

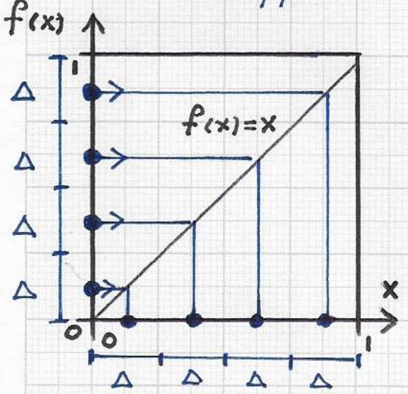
• Laplacian eigenfunctions and neural networks:... ● Thus, one must be interested how well such a stochastic iterated Richardson extrapolation scheme estimates the "exact" density value ϱ at a specific location when only discrete sample sets are given. To gain some level of insight, one can generate (high-resolution) discrete sample sets via a simple analytically defined distribution function and subsequently use the samples – and Richardson extrapolation – to see whether it is possible to reconstruct the original analytical distribution/density function.



• Examples. We consider several examples of increasing complexity to explore the "density reconstruction" issue. The left figures illustrate the "simplest" distribution function: $f(x) = x$. We use $N = 12$ intervals of length $\Delta = 1/12$ to subdivide the x-interval $[0, 1]$ with 13 equidistantly placed locations x_i. We want to estimate the value of $\varrho(*)$. We place '*' at the center of three x-axis intervals with widths $4/6$, $2/6$ and $1/6$. By counting the numbers of samples in the intervals one obtains ϱ_0, ϱ_1 and ϱ_2 – all having the value 12. ...

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• **Laplacian eigenfunctions and neural networks...** If one assigned a "mass" of $1/12$ to the 11 interior points x_i and a "mass" of $1/24$ to the two endpoints x_0 and x_{12}, then the "total mass" would be $\frac{1}{12} \cdot 11 + \frac{1}{24} \cdot 2 = 1$. This fact simply reflects that an f-interval of length ("mass") 1 is mapped to an x-interval of length ("mass") 1. One can also associate the individual "mass points" with the centers of intervals of uniform length Δ – as shown in the left figure – instead of viewing the endpoints of intervals as "mass points."



In the example shown in the left figure, each point has a "mass" of $1/4$, and the "total mass" is $\frac{1}{4} \cdot 4 = 1$.

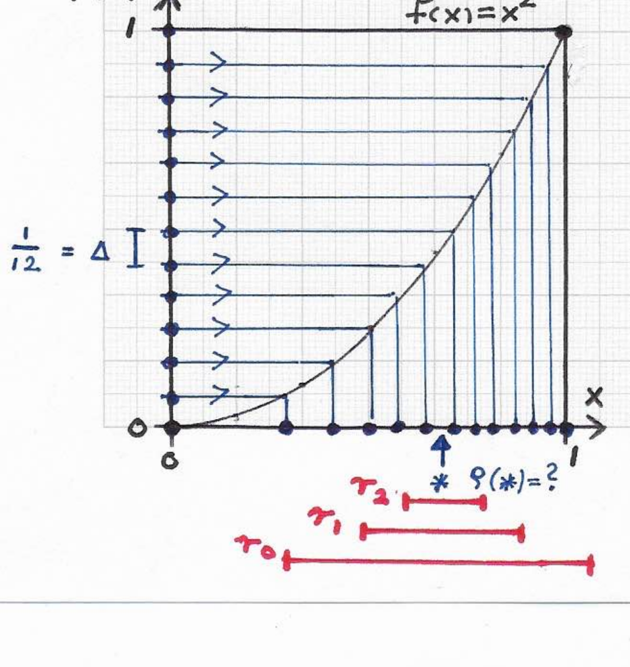
⇒ Considering the example illustrated on the previous page again and using these "normalized masspoints," one obtains the three local density values $\rho_2 = 2 \cdot \left(\frac{1}{12} : \frac{1}{6}\right) = 1$, $\rho_1 = 4 \cdot \left(\frac{1}{12} : \frac{1}{3}\right) = 1$ and $\rho_0 = 8 \cdot \left(\frac{1}{12} : \frac{2}{3}\right) = 1$. Thus, the two resulting (linear) extrapolations are $A_0^1 = (A_1^0 - A_0^0/2) : \frac{1}{2}$ $= 2(A_1^0 - A_0^0/2) = 2(\rho_1 - \rho_0/2) = 1$ and $A_1^1 = (A_2^0 - A_1^0/2) : \frac{1}{2}$ $= 2(A_2^0 - A_1^0/2) = 2(\rho_2 - \rho_1/2) = 1$. Iterating Richardson extrapolation once, one obtains $A_0^2 = (A_1^1 - A_0^1/4) : \frac{3}{4}$ $= \frac{4}{3}(A_1^1 - A_0^1/4) = \frac{4}{3}(1 - 1/4) = 1$. For completeness, the resulting table of A_i^j-values is shown here. In this "constant-value case," iteration is not really necessary.

$i \backslash j$	0	1	2
0	$\rho_0 = A_0^0 = 1$	$A_0^1 = 1$	$A_0^2 = 1$
1	$\rho_1 = A_1^0 = 1$	$A_1^1 = 1$	
2	$\rho_2 = A_2^0 = 1$		

...

OBJECT AND MATERIAL EIGENFUNCTIONS - Cont'd.

• Laplacian eigenfunctions and neural networks:... It is important to realize that the distribution function $f(x)$ and the density to be computed for points / locations on the x-axis are related via f's first derivative: $f'(x) = \frac{d}{dx} f$. Using the calculus of differences and quotients of differences — to establish the link to a discrete, finite difference setting — we consider the difference formula $f' = \Delta y / \Delta x$. In our first example, we have used the distribution function $f(x) = x$. **This function's derivative is $f' = 1$, and the density calculated via Richardson extrapolation from discrete samples on the x-axis is $\rho = 1$.** Thus, it can be assumed that density estimation is related to derivative estimation — and that one can potentially understand and optimize stochastic iterated Richardson extrapolation for use in derivative approximation. We consider a few more examples to explore this aspect.



We explore the quadratic distribution function $f(x) = x^2$, using $N = 12$ f-intervals of length $\Delta = 1/12$. We use f_i-values $f_i = i/12$, $i = 0 \ldots 12$, and calculate the corresponding x_i-values, i.e., $x_i = \sqrt{f_i} = \sqrt{3i}/6$. The left figure illustrates this example.

...

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks:... To be complete, we provide the specific values involved in this example.

i	0	1	2	3	4	5	6	7	8	9	10	11	12	13
f_i	0	$1/12$	$1/6$	$1/4$	$1/3$	$5/12$	$1/2$	$7/12$	$2/3$	$3/4$	$5/6$	$11/12$	1	$13/12$
x_i	0	$\sqrt{3}/6$	$\sqrt{6}/6$	$1/2$	$\sqrt{3}/3$	$\sqrt{15}/6$	$\sqrt{2}/2$	$\sqrt{21}/6$	$\sqrt{6}/3$	$\sqrt{3}/2$	$\sqrt{30}/6$	$\sqrt{33}/6$	1	...
	0	.29	.41	.5	.58	.65	.71	.76	.82	.87	.91	.96	1	...

x_i-values inside intervals with lengths r_0, r_1, r_2

Once again, the objective is the estimation of $\varrho(*)$-values for points/locations on the x-axis – not knowing the original distribution function and having to approximate the value of $\varrho(*)$ via finite sample set based extrapolation. For normalization purposes, we view all (interior) points on the f-axis as "normalized mass points" with a "mass" of $1/12$. Via the distribution function (unknown!) these "masses" generate the observed non-uniform "mass/masspoint" distribution – and thus density – on the x-axis. We must calculate initial density values $\varrho_0 = \varrho_0(*)$, $\varrho_1 = \varrho_1(*)$ and $\varrho_2 = \varrho_2(*)$ by determining the numbers of "masspoint" samples in the interior of x-axis intervals of lengths r_0, r_1 and r_2 – all having $*$ at their common center. Considering the table provided above, we use $* = .63$ and $r_0 = .72$, $r_1 = .36$ and $r_2 = .18$.

i	0	1	2
r_i	.72	.36	.18
ϱ_i	1.30952	1.19048	1.42857

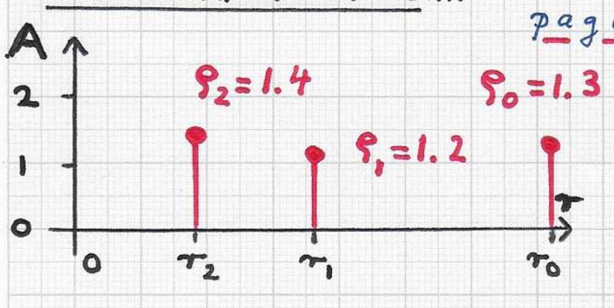
Initial density estimates $\varrho_i(*) = A_i^0$.

The interval lengths r_0, r_1 and r_2 have these associated x-intervals: r_0: [.28, .98]; r_1: [.455, .805]; r_2: [.5425, .7175].

...

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks:...



The top table provided on the previous page indicates via red line segments under the table which x_i-values lie inside the x-axis intervals associated with interval lengths r_0, r_1 and r_2.

The values of ρ_i, $i=0...2$, are calculated as follows:

$\rho_0 = 11 \cdot (\frac{1}{12} : .7) = 1.30952$ (11 x_i-values inside interval);

$\rho_1 = 5 \cdot (\frac{1}{12} : .35) = 1.19048$ (5 x_i-values inside interval);

$\rho_2 = 3 \cdot (\frac{1}{12} : .175) = 1.42857$ (3 x_i-values inside interval).

Iterated Richardson extrapolation yields the following results:

$A_0^1 = (A_1^0 - A_0^0/2) : \frac{1}{2} = 2(\rho_1 - \rho_0/2) = 2(1.19048 - .65476) = 1.07144$;

$A_1^1 = (A_2^0 - A_1^0/2) : \frac{1}{2} = 2(\rho_2 - \rho_1/2) = 2(1.42857 - .59524) = 1.66726$;

$A_0^2 = (A_1^1 - A_0^1/4) : \frac{3}{4} = \frac{4}{3}(1.66726 - .26786) = 1.86587$.

i \ j	0	1	2
0	1.30952	1.07144	1.86587
1	1.19048	1.66726	
2	1.42857		

The table (left) summarizes the A_i^j-values for this quadratic distribution function example. The "analytically correct value" for $\rho(0.63)$ is $\frac{d}{dx} f(0.63) = f'(0.63) = 2 \cdot 0.63 = 1.26$. The fact that the values $A_0^2(0.63)$ and $f'(0.63)$ are rather different is a consequence of the small number of samples ($N=13$) used for extrapolation. Specifically, the value of ρ_2 is only a "poor approximation," and this fact has an impact on all subsequent extrapolated values depending on ρ_2.

...