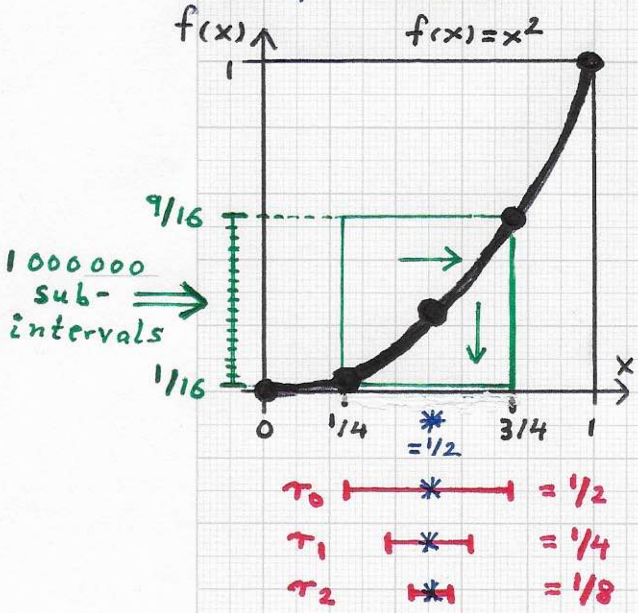
## ■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks:... Thus, it is desirable to gain more insight in the relationship between the number $N$ of sample points, used to represent a certain density function via a discrete, finite point set on the x-axis, and the approximation quality/error of iterated Richardson extrapolation. We consider the example sketched in the left figure.




The goal is to estimate a density value for $x = * = 1/2$. We will determine the numbers of samples in x-axis intervals of total lengths $r_0 = 1/2$, $r_1 = 1/4$ and $r_2 = 1/8$, with all intervals having $x = 1/2$ as common midpoint. The (unknown) distribution function $f$ is $f(x) = x^2$; the f-axis interval $[1/16, 9/16]$ is uniformly subdivided into 1000000 subintervals. Thus, subinterval endpoints on the f-axis are

$$f_i = \frac{1}{16} + i\,\frac{\Delta}{2} = (1 + 8i\Delta)/16\,,\quad i = 0 \ldots 1000000 \text{ and } \Delta = 1/1000000.$$

The corresponding $x_i$-values in the $r_0$-interval $[1/4, 3/4]$ are

$$x_i = \sqrt{f_i} = \frac{1}{4}\sqrt{1+8i\Delta}\,,\quad i = 0 \ldots 1000000.$$

We must determine the i-value that maps an $f_i$-value to $x_i = 1/2 = *$. We obtain this i-value as follows:

$$\frac{1}{2} = \frac{1}{4}\sqrt{1+8i\Delta} \Rightarrow 2 = \sqrt{1+8i\Delta} \Rightarrow 4 = 1 + 8i\Delta \Rightarrow i = \frac{3}{8\Delta}$$

$$\Rightarrow i = 3000000/8 = 375000.$$ **Thus, 375000 x-samples lie in $[1/4, 1/2]$ and 625000 lie in $[1/2, 3/4]$.**

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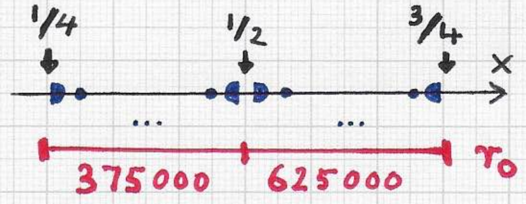
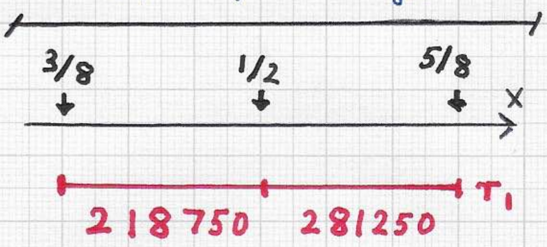
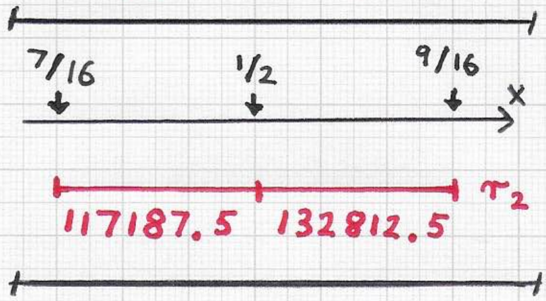
## ■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks: ...




The symbols '◗' and '◖' designate samples "at the ends" that count as "half-samples" only.




Thus, the value of $\varrho_2$ is

$$\varrho_2 = (117187.5 + 132812.5) \cdot \left(\frac{1}{1000000} : \frac{1}{8}\right) = 2.$$

In summary, all density estimates are equal, i.e.,

$$\varrho_2 = \varrho_1 = \varrho_0 = \varrho(1/2) = 2.$$

Consequently, all $A_i^j$-values are 2.

This distribution of $x_i$-samples – 375000 : 625000 – is sketched in the left figure. The value of $\varrho_0$ associated with $\tau_0$ is

$$\varrho_0 = 1000000 \cdot \left(\frac{1}{1000000} : \frac{1}{2}\right) = 2.$$

Next, one must determine the $i$-values that are mapped to specific $x$-values in the $\tau_1$-interval. The left figure shows this case. One obtains these values:

$$\frac{3}{8} = \frac{1}{4}\sqrt{1+8i\Delta} \Rightarrow \dots \Rightarrow i = 156250;$$

$$\frac{5}{8} = \frac{1}{4}\sqrt{1+8i\Delta} \Rightarrow \dots \Rightarrow i = 656250.$$

The following numbers result – the numbers of samples in [3/8, 1/2] and in [1/2, 5/8]:

$$375000 - 156250 = 218750;$$

$$656250 - 375000 = 281250.$$

The value of $\varrho_1$ for $\tau_1$ is

$$\varrho_1 = (218750 + 281250) \cdot \left(\frac{1}{1000000} : \frac{1}{4}\right) = 2.$$

For the $\tau_2$-interval one obtains:

$$7/16 = 1/4\sqrt{1+8i\Delta} \Rightarrow \dots \Rightarrow i = 257812.5$$

$$9/16 = 1/4\sqrt{1+8i\Delta} \Rightarrow \dots \Rightarrow i = 507812.5$$

The numbers of samples in [7/16, 1/2], [1/2, 9/16] are:

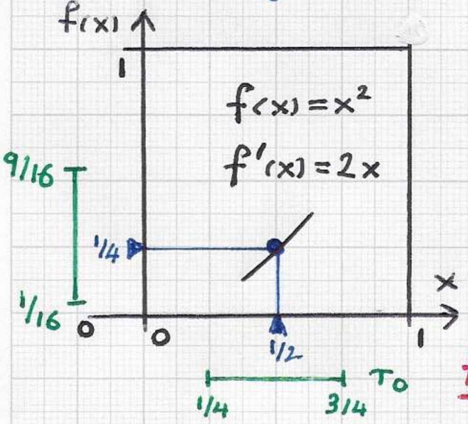
$$375000 - 257812.5 = 117187.5;$$

$$507812.5 - 375000 = 132812.5.$$

...

## ■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks:... • **Note.** It is relevant to consider the differential quotient in the neighborhood of $x = 1/2$ — to understand how local change in $f_i$-values relates to differential behavior (and potentially to local density value – or not). The left figure sketches the situation for $f(x)=x^2$. Based on the 1000000 sub-intervals used in the described scenario, one can consider these local values:




$x_{374999} = .4999995$; $x_{375001} = .5000005$;
$f_{374999} = .07421871875$; $f_{375001} = .07421878125$.

These four local x- and f-values define a local differential quotient: $\Delta x = 0.000001$, $\Delta f = 0.0000000625$ and $\Delta f / \Delta x = 1/16$. The actual first derivative value of $f(x)=x^2$ at $x = 1/2$ is $f'(1/2) = 1$. Thus, it is not immediately obvious how f's derivative relates to density (of samples on the x-axis). Therefore, one could consider the potential relationship of f and density based on integral behavior.

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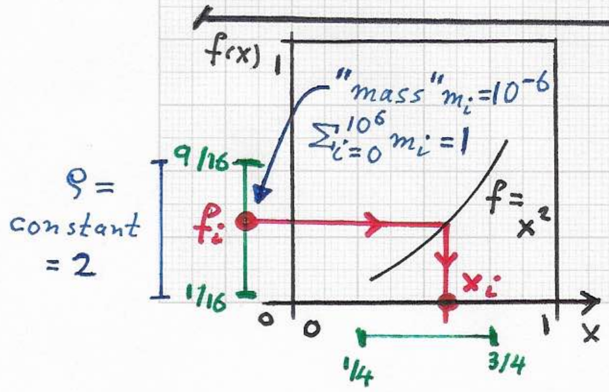


It is helpful to adopt a "physical view": The f-interval $[1/16, 9/16]$ is subdivided into 1000000 equal-length sub-intervals of length $1/1000000$ — being the "mass" of each f-sub-interval. Thus, the total "mass" of f-interval $[1/16, 9/16]$ is 1, and the constant "density" value is $1/(9/16 - 1/16) = 1/(1/2) = 2$.

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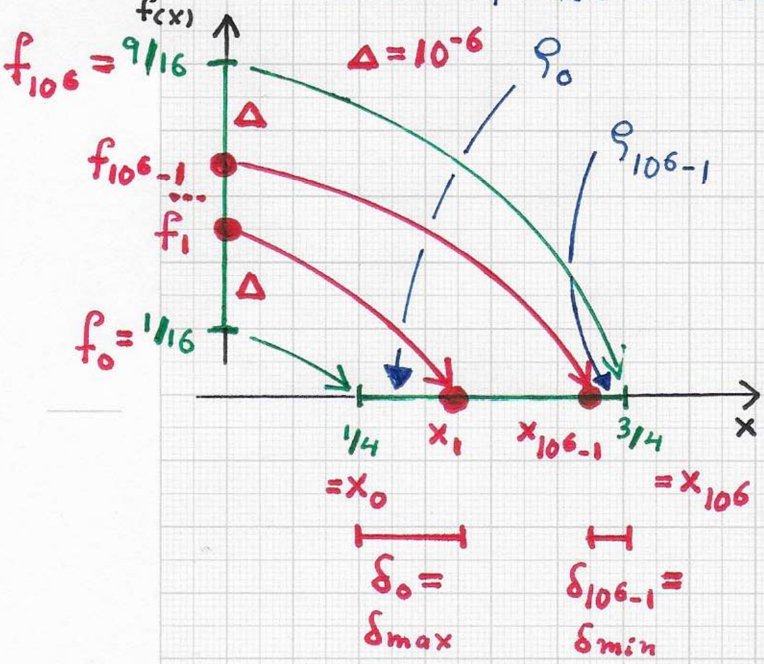
## ■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks:... One can explore the variation of "physical density" between $\frac{1}{4}$ and $\frac{3}{4}$ on the x-axis by calculating the density values at the ends of the x-interval $[1/4, 3/4]$. The uniformly placed $f_i$-values on the f-axis ($\Delta = f_{i+1} - f_i = \text{const} = 10^{-6}$, $i = 0 \ldots (10^6 - 1)$) are mapped to non-uniformly placed $x_i$-values on the x-axis ($\delta_i = x_{i+1} - x_i$, $i = 0 \ldots (10^6-1)$). For this specific example, the numerical values needed for density values of the x-axis intervals $[\frac{1}{4}, x_1]$ and $[x_{10^6-1}, \frac{3}{4}]$




are: $x_0 = 1/4$, $x_1 = 1/4\sqrt{1 + 8 \cdot 10^{-6}} = 0.25000099999$

$$\Rightarrow \underline{\delta_{max}} = \delta_0 = x_1 - x_0 = \underline{0.00000099999}.$$

$$x_{10^6} = 3/4,\ x_{10^6-1} = 1/4\sqrt{1 + 8 \cdot (10^6 - 1) \cdot 10^{-6}}$$

$$= 1/4\sqrt{1 + 8(1 - 10^{-6})} = 0.74999966666$$

$$\Rightarrow \underline{\delta_{min}} = \delta_{10^6-1} = x_{10^6} - x_{10^6-1} = \underline{0.00000033334}.$$

The resulting density values "mass/volume" $\varrho_0$ and $\varrho_{10^6-1}$ are:

$$\underline{\varrho_0} = \Delta / \delta_{max} = 10^{-6} / 0.00000099999 = \underline{1.0000100001}$$

$$\underline{\varrho_{10^6-1}} = \Delta / \delta_{min} = 10^{-6} / 0.00000033334 = \underline{2.9999400012}.$$

• **The important insight is the fact that these two "local sub-interval densities" differ by a factor of 3.**

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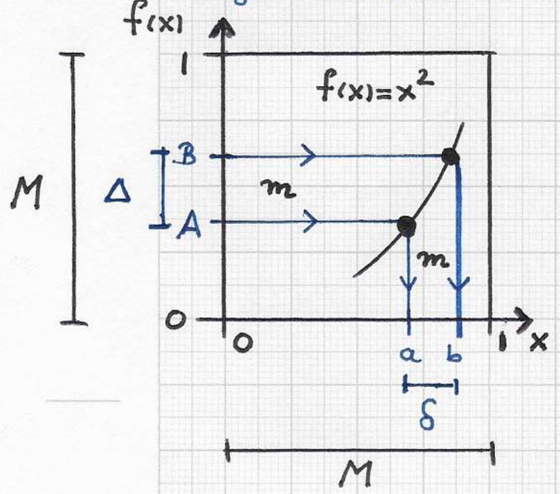
## OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

- Laplacian eigenfunctions and neural networks:... To explore the variation of density on the x-axis (the x-interval $[0,1]$) in more detail and ANALYTICALLY, we discuss the general setting. A total mass of $M$ is distributed uniformly on the f-axis (the f-interval $[0,1]$), i.e., the density value on the f-axis is constant: $\rho = M/1 = M$. Since density is constant on the f-axis, the mass $m$ associated with an interval $[A,B]$ of length $\Delta = B-A$ is $m = \Delta \cdot M$. Via the distribution function $f(x) = x^2$, this mass $m$ is now "mapped" to the x-axis interval $[a,b]$, where $a = \sqrt{A}$ and $b = \sqrt{B}$; the length ("1D volume") of this interval is $\delta = b-a = \sqrt{B}-\sqrt{A}$.




Thus, the resulting density for this x-axis interval is $\bar{\rho} = m/\delta = m/(b-a) = m/(\sqrt{B}-\sqrt{A})$. Using the fact that $m = \Delta \cdot M$, one obtains

$$\bar{\rho} = M \cdot (B-A)/(\sqrt{B}-\sqrt{A})$$
$$\Leftrightarrow \bar{\rho} = M \cdot \Delta/\delta .$$

(See figure.)

- Example. We consider two extreme cases:

i) $A=0,\ B=1/10000$: $a=\sqrt{A}=0,\ b=\sqrt{B}=1/100$

$$\Rightarrow \bar{\rho} = M \cdot (1/10000)/(1/100) = M \cdot 1/100 .$$

ii) $A = 9999/10000,\ B=1$: $a = \sqrt{9999}/100,\ b=1$

$$\Rightarrow \bar{\rho} = M \cdot (1-9999/10000)/(1-\sqrt{9999}/100)$$
$$= M \cdot 1.99994999875 .$$

...