Bernd Hamann
Stratovan

9/26/2022

■ OBJECT AND MATERIAL EIGENFUNCTIONS - Cont'd.

• **Laplacian eigenfunctions and neural networks:...** Defining $\varepsilon = B - A$, with $\varepsilon \ll 1$, one obtains the more general expressions for these two extreme cases:

i) $A = 0$, $B = \varepsilon$: $a = 0$, $b = \sqrt{\varepsilon}$

$$\Rightarrow \bar{\varrho} = M \cdot (\varepsilon - 0)/(\sqrt{\varepsilon} - 0) = M \cdot \varepsilon / \sqrt{\varepsilon} = M\sqrt{\varepsilon}.$$

ii) $A = 1 - \varepsilon$, $B = 1$: $a = \sqrt{1-\varepsilon}$, $b = 1$

$$\Rightarrow \bar{\varrho} = M \cdot (1 - (1-\varepsilon))/(1 - \sqrt{1-\varepsilon}) = M \cdot \varepsilon / (1 - \sqrt{1-\varepsilon})$$
$$= M \cdot (\varepsilon \cdot (1 + \sqrt{1-\varepsilon}))/((1 - \sqrt{1-\varepsilon})(1 + \sqrt{1-\varepsilon}))$$
$$= M \cdot (1 + \sqrt{1-\varepsilon})$$

Thus, one can calculate the $\bar{\varrho}$-values for $\varepsilon \to 0$:

i) $\bar{\varrho} = M\sqrt{\varepsilon} \Rightarrow \lim_{\varepsilon \to 0} \bar{\varrho} = M \cdot 0 = 0$.

ii) $\bar{\varrho} = M(1 + \sqrt{1-\varepsilon}) \Rightarrow \lim_{\varepsilon \to 0} \bar{\varrho} = M(1+1) = 2M$.

The value of $\bar{\varrho}$ for the general case $(B - A) \to 0$ can be calculated as follows:

$$\Delta = B - A = (A + \varepsilon) - A = \varepsilon,$$
$$\delta = \sqrt{B} - \sqrt{A} = \sqrt{A+\varepsilon} - \sqrt{A},$$
$$\bar{\varrho} = M \cdot \Delta/\delta = M \cdot \varepsilon / (\sqrt{A+\varepsilon} - \sqrt{A})$$
$$= M \cdot \varepsilon (\sqrt{A+\varepsilon} + \sqrt{A}) / ((\sqrt{A+\varepsilon} - \sqrt{A})(\sqrt{A+\varepsilon} + \sqrt{A}))$$
$$= M \cdot (\sqrt{A+\varepsilon} + \sqrt{A}).$$

$$\Rightarrow \lim_{\varepsilon \to 0} \bar{\varrho} = 2M\sqrt{A}.$$

Alternatively, one can set $\Delta = B - (B - \varepsilon) = \varepsilon$ etc., and the results are: $\bar{\varrho} = M \cdot (\sqrt{B} + \sqrt{B - \varepsilon})$,

$$\lim_{\varepsilon \to 0} \bar{\varrho} = 2M\sqrt{B}.$$

...

■ **OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.**

• Laplacian eigenfunctions and neural networks:... MOST IMPORTANTLY, FOR $\varepsilon \Rightarrow 0$ $A = B$, I.E., THE INTERVAL $[A, B]$ IS OF LENGTH 0. THUS, ONE OBTAINS A DENSITY VALUE FOR $f = A = B$. THE RESULTING x-AXIS DENSITY VALUE IS

$$\bar{\rho}(x) = \bar{\rho}(\sqrt{f}) = 2M\sqrt{f} \;.$$

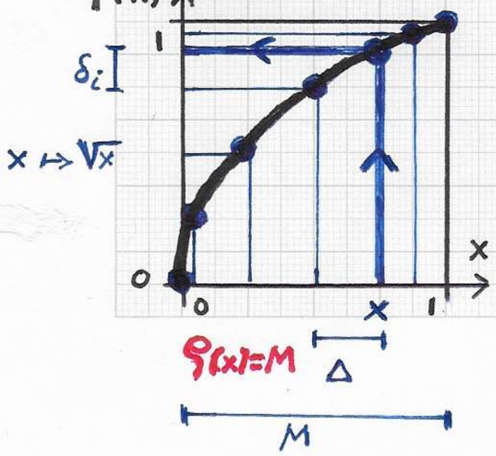
!

Knowing this relationship between x, f, $\rho\,(=M)$ and $\bar{\rho}$, one can employ a simple substitution to describe the "Law" in an easier way:

$$M \cdot f \mapsto 2M \cdot \sqrt{f} \qquad \mid \text{ substitution: } X^2 = f$$

$$\Leftrightarrow \; M \cdot X^2 \mapsto 2M \cdot \sqrt{X^2} = 2M \cdot X$$

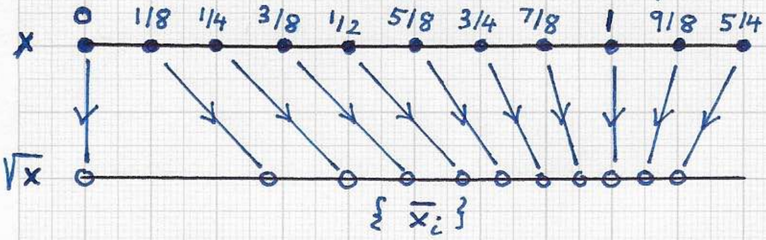
Thus, the function $F(X) = M \cdot X^2$ is mapped to the function $F'(X) = \frac{d}{dX} F = 2M \cdot X$ (in this case and generally). Of course, one can also use the horizontal axis, the x-axis, to represent the axis with uniformly distributed "mass" (constant density M) associated with it — and, vice versa, use the vertical axis, the f-axis, to represent the axis that the "mass" of the x-axis is distributed onto. The left figure sketches this alternate viewpoint: A total "mass" M is associated with the x-axis interval $[0, 1]$ and uniformly distributed onto equal-length sub-intervals of length Δ. Sub-intervals of length Δ are mapped to variable-length corresponding f-axis intervals of lengths δ_i via the mapping $f(x) = \sqrt{x}$.



...

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks: … • **Note.** Another, possibly more effective, way to visualize the mapping of a constant, uniform density to a varying, non-uniform density places the domain spaces next to each other, indicating how points are mapped from one domain (with uniform density) to the other domain, the "deformed domain space":



Simple mapping of equidistantly placed points $x_i = i/8$, $i=0,\ldots$, to corresponding points $\bar{x}_i = \sqrt{x_i}$, $i = 0, \ldots$. The approximate $\bar{x}_i$-values are:

$$\bar{x}_0, \bar{x}_1, \bar{x}_2, \ldots = 0, \sqrt{2}/4, 1/2, \sqrt{6}/4, \sqrt{2}/2, \sqrt{10}/4, \sqrt{3}/2, \sqrt{14}/4, 1, 3\sqrt{2}/4, \sqrt{5}/2, \ldots$$

$$\approx 0, .35, .5, .61, .71, .79, .87, .94, 1, 1.06, 1.12, \ldots$$

Assuming again that a total "mass" M is associated with the x-interval $[0,1]$, the constant x-axis density value is $\rho = M/1 = M = m/\Delta = M/8 : 1/8 = M/8 \cdot 8 = M$, where $m = \frac{M}{8}$, $\Delta = \frac{1}{8}$.

The density associated with an $\bar{x}$-axis interval $[\bar{x}_i, \bar{x}_{i+1}]$ is:

$$\bar{\rho}_i = m / \delta_i = m / (\sqrt{x_{i+1}} - \sqrt{x_i}) = m / (\sqrt{(i+1)/8} - \sqrt{i/8})$$

$$= 2m\sqrt{2} / (\sqrt{i+1} - \sqrt{i}) = 2m\sqrt{2}\,(\sqrt{i+1} + \sqrt{i}) / ((\sqrt{i+1} - \sqrt{i})(\sqrt{i+1} + \sqrt{i}))$$

$$= 2m\sqrt{2}\,(\sqrt{i+1} + \sqrt{i}) = \frac{M}{4}\sqrt{2}\,(\sqrt{i+1} + \sqrt{i}) \; .$$

This discrete view, using specific points x_i and $\bar{x}_i$, can also be changed to the continuum-space view. Further, one could then employ an even more general description based on a continuous deformation of space.

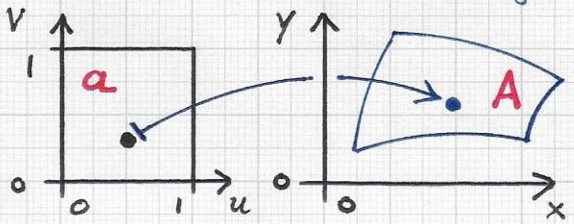
…

Bernd Hamann
Stratovan

9/29/2022

■ OBJECT AND MATERIAL EIGENFUNCTIONS - Cont'd.

• Laplacian eigenfunctions and neural networks:... • **Note.** "Deformation of space" is, for example, a topic in vector calculus, see "Vector Calculus" by Jerrold Marsden and Anthony Tromba. We briefly summarize the few concepts that we need for the purpose of density functions — which require one to understand the relationship between the amount of "mass" associated with a deformed region of space — 1D, 2D, 3D, ..., N-dimensional space. Concepts can be explained simply for the 2D case. The figures sketch the scenario where a



unit square in uv-parameter space is deformed/mapped to a corresponding 4-sided region in xy-image space.

The deformation maps a tuple $\mathbf{u} = (u, v)$ to $\mathbf{x} = (x, y)$, where $\mathbf{x} = \mathbf{x}(\mathbf{u}) = (x(u,v), y(u,v))$. **The Jacobian determinant of this mapping is defined as**

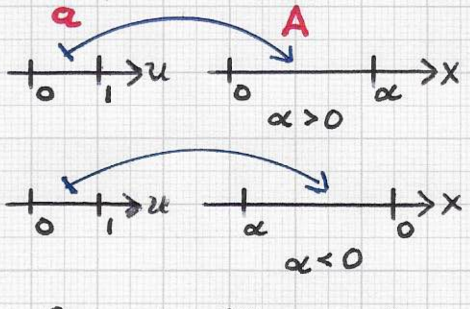
$$J = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}.$$

The domain area a is mapped to the area A. The area A is obtained via integration of J over the uv-domain square, i.e., $A = \int_{[0,1]^2} J \, du\, dv$. (The value of A is signed; the sign indicates orientation.)

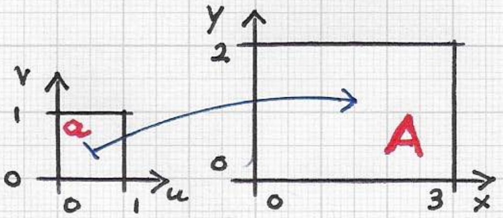
•••

OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks:...



Simple 1D example.



Simple 2D example.

The example shown in the left figures (top) is based on the simple mapping (scaling) $x(u) = \alpha u$, where α can be positive or negative. Thus, the unit interval, of length $a = 1$, is mapped to the interval with length A, where $A = \int_0^1 \alpha \, du = \alpha$. The 2D non-uniform scaling shown in the left figure uses the linear mapping $\mathbb{X}(\mathbf{u}) = (3u, 2v)$. Thus, the Jacobian determinant J has the value $J = 3 \cdot 2 - 0 \cdot 0 = 6$. The area A is therefore $A = \int_0^1 \int_0^1 6 \, du \, dv = 6$. We can now utilize this method for calculating the A-value for our 1D example $\bar{x}(x) = \sqrt{x} = x^{1/2}$. In this case, we obtain $J = |\bar{x}_x| = \frac{1}{2} x^{-1/2}$; mapping the interval $[x_i, x_{i+1}]$, for example, generates the image interval that has the length

$$A = \int_{x_i}^{x_{i+1}} \frac{1}{2} x^{-1/2} dx = \frac{1}{2} \int_{x_i}^{x_{i+1}} x^{-1/2} \, dx = \sqrt{x} \,\Big|_{x_i}^{x_{i+1}}$$

$$= \sqrt{x_{i+1}} - \sqrt{x_i} \, .$$

Thus, the application of the Jacobian determinant-based formula to this simple $x \mapsto \sqrt{x}$ example correctly yields the value for A, i.e., $\sqrt{x_{i+1}} - \sqrt{x_i}$.

...