■ OBJECT AND MATERIAL EIGENFUNCTIONS - Cont'd.

• Laplacian eigenfunctions and neural networks:...

In this example, the two transformation matrices T and $\tilde{T}$ are

$T = RD^{-1} = \begin{bmatrix} 1/2 & 1/2 \\ 0 & 1 \end{bmatrix}$ and $\tilde{T} = DR^{-1} = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}$. In this case, it is not immediately obvious how to use T and/or $\tilde{T}$ to perform a desirable correction of the initial classification prediction.

We now consider a 3-class classification scenario:

TRUE stream: 012012012

Classification as: 120120120

$\Rightarrow R = \begin{bmatrix} 0 & 3 & 0 \\ 0 & 0 & 3 \\ 3 & 0 & 0 \end{bmatrix}$, $R^{-1} = \begin{bmatrix} 0 & 0 & 1/3 \\ 1/3 & 0 & 0 \\ 0 & 1/3 & 0 \end{bmatrix}$,

$D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, $D^{-1} = \begin{bmatrix} 1/3 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}$, $T = RD^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$, $\tilde{T} = DR^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.

Classes 0, 1 and 2 are identified with $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

(*) We apply T to these three column vectors and obtain:

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \cdot \left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right] = \left[\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right]$$

. The interpretation of this result is the following:

- A pre-classification '0' should be corrected to '2'.
- " " '1' " " " " '0'.
- " " '2' " " " " '1'.

Using these individual corrections, the initial pre-classification result stream 120120120 becomes the optimal correction stream 012012012.

(*) In the example on pages 9-10, we also could have used T, since $T = \tilde{T}$ in that case.

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OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks... In a testing-validation phase of the classification system, with known ground truth information, one can observe and calculate the matrices R, D, R^{-1}, D^{-1}, T and $\tilde{T}$. We will now discuss several examples to understand how these matrices can be used to improve an initial pre-classification stream such that a final correction stream is (at least) statistically optimal. (•Note: It is important to consider that the specific order used to stream a set of materials/objects cannot be included in the analysis of classification performance of a data stream; the analysis must be understood as a statistical characterization of the system.) We explore examples:

TRUE stream: 00001111 ⇒ $R = \begin{bmatrix} 2 & 2 \\ 0 & 4 \end{bmatrix}$, $D^{-1} = \begin{bmatrix} 1/4 & 0 \\ 0 & 1/4 \end{bmatrix}$,

Classification as: 00111111

$T = RD^{-1} = \begin{bmatrix} 1/2 & 1/2 \\ 0 & 1 \end{bmatrix}$. The column vectors $\begin{pmatrix} 1/2 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1/2 \\ 1 \end{pmatrix}$ result when applying T to the vectors representing class 0 and class 1, i.e., the vectors $(1,0)^T$ and $(0,1)^T$, respectively. Thus, the meaning of T is:

- The pre-classification result '0' should statistically be mapped to (statistically) corrected final result values/classes '0' and '1' using the ratio $\frac{1}{2}:0 \triangleq 100:0$; i.e., '0' always remains '0'.

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■ OBJECT AND MATERIAL EIGENFUNCTIONS - Cont'd.

• Laplacian eigenfunctions and neural networks:...

• The pre-classification result '1' should statistically be mapped to (statistically) corrected final result values/classes '0' and '1' using the ratio $\frac{1}{2}:1 \hat{=} 33.3:66.7$; i.e., a '1' should be replaced by '0' with a probability of 0.333 and should remain '1' with a probability of 0.667.

TRUE stream : 00000001
Classification as : 11111110 $\Rightarrow R = \begin{bmatrix} 0 & 7 \\ 1 & 0 \end{bmatrix}$, $D^{-1} = \begin{bmatrix} 1/7 & 0 \\ 0 & 1 \end{bmatrix}$,

$T = RD^{-1} = \begin{bmatrix} 0 & 7 \\ 1/7 & 0 \end{bmatrix}$. The meaning of T's column vectors is:

• The pre-classification result '0' should be mapped to classes '0' and '1' using the ratio $0:\frac{1}{7} \hat{=} 0:100$; i.e., '0' should always be replaced by '1'.

• The pre-classification result '1' should be mapped to classes '0' and '1' using the ratio $7:0 \hat{=} 100:0$; i.e., '1' should always be replaced by '0'.

TRUE stream : 00001111
Classification as : 01110000 $\Rightarrow R = \begin{bmatrix} 1 & 3 \\ 4 & 0 \end{bmatrix}$, $D^{-1} = \begin{bmatrix} 1/4 & 0 \\ 0 & 1/4 \end{bmatrix}$,

$T = RD^{-1} = \begin{bmatrix} 1/4 & 3/4 \\ 1 & 0 \end{bmatrix}$. The meaning of T is:

• The pre-classification result '0' should be mapped to classes '0' and '1' using the ratio $\frac{1}{4}:1 \hat{=} 25:100$; i.e., a '0' should remain a '0' with a probability of 0.2 and should be replaced by '1' with a probability of 0.8.

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■ OBJECT AND MATERIAL EIGENFUNCTIONS - Cont'd.

• Laplacian eigenfunctions and neural networks:...

• The pre-classification result '1' should be mapped to classes '0' and '1' using the ratio $\frac{3}{4}:0 \hat{=} 100:0$; i.e., a '1' should always be replaced by '0'.

(• **Note.** One can also define a matrix $\hat{T}$ that, when applied to the class-0 vector and class-1 vector, i.e., to $(1,0)^T$ and $(0,1)^T$, produces the "statistically correct" manipulation of the pre-classification stream 01110000. By simple comparison of this stream with the true stream 00001111, one can determine the following necessary "statistical corrections": '0': in 1 of 5 cases, replace '0' by '0';
in 4 of 5 cases, replace '0' by '1'.
'1': in 3 of 3 cases, replace '1' by '0';
in 0 of 3 cases, replace '1' by '1'.
The matrix $\hat{T}$ is thus defined as $\hat{T} = \begin{bmatrix} 1/5 & 3/3 \\ 4/5 & 0/3 \end{bmatrix}$.)

We can also explore the "statistical correction" of this pre-classification stream symbolically. The matrices are:

$$R = \begin{bmatrix} r_{0,0} & r_{0,1} \\ r_{1,0} & r_{1,1} \end{bmatrix}, \quad D = \begin{bmatrix} n_0 & 0 \\ 0 & n_1 \end{bmatrix}, \quad D^{-1} = \begin{bmatrix} 1/n_0 & 0 \\ 0 & 1/n_1 \end{bmatrix}, \quad T = \begin{bmatrix} r_{0,0}/n_0 & r_{0,1}/n_1 \\ r_{1,0}/n_0 & r_{1,1}/n_1 \end{bmatrix}.$$

Here, the values of n_0 and n_1 are the numbers of class-0 and class-1 occurrences in the ground truth stream, i.e., $n_0 = 4$ and $n_1 = 4$. For our last example, we obtain $T = \begin{bmatrix} 1/4 & 3/4 \\ 4/4 & 0/4 \end{bmatrix}$. Again, the replacement ratios are 1:4 and 3:0.

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■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks: ... Another very important aspect concerning the characterization of classification performance via such a matrix approach concerns the convergence behavior / stability and changing behavior / evolution "over time." In fact, material characteristics (and thus features) and/or scanning system characteristics (and thus images of the same material class) can change over time. This fact, in turn, can have a minor or major impact on the performance matrices R, T etc. For example, we can compute the sequence of R matrices that results when considering 1, 2, 4, 8, 16 and 32 class indices in a 3-class class index stream. We use the 32-index stream from page 3 (11/17/2022); the first stream is the ground truth, the second stream is the classification stream, i.e., the pre-classification:

$$\begin{array}{c|c|cc|cccc|cccccccc|cccccccccccccccc|}
0&0&0&0&1&0&0&2&0&1&0&0&0&1&1&0&0&2&2&2&0&0&0&0&2&0&2&0&0&2&0&2\\
0&0&0&1&0&0&2&0&0&1&0&2&0&1&1&1&0&0&2&2&0&0&0&1&2&0&0&0&0&1&0&0
\end{array}_{32}$$

(1, 2, 4, 8, 16)

We call the resulting R matrices ${}^{1}R, {}^{2}R, {}^{4}R, {}^{8}R, {}^{16}R$ and ${}^{32}R$; they are defined by the comparison of 1, 2, 4, 8, 16 and 32 index pairs in the shown index streams:

$${}^{1}R = \begin{bmatrix} 1&0&0\\0&0&0\\0&0&0 \end{bmatrix},\ {}^{2}R = \begin{bmatrix} 2&0&0\\0&0&0\\0&0&0 \end{bmatrix},\ {}^{4}R = \begin{bmatrix} 3&1&0\\0&0&0\\0&0&0 \end{bmatrix},\ {}^{8}R = \begin{bmatrix} 4&1&1\\1&0&0\\1&0&0 \end{bmatrix},\ {}^{16}R = \begin{bmatrix} 7&2&2\\1&3&0\\1&0&0 \end{bmatrix},\ {}^{32}R = \begin{bmatrix} 15&3&2\\1&3&0\\4&1&3 \end{bmatrix}.$$

One sees that this R matrix sequence converges slowly.

Considering ${}^{32}R$, we have ${}^{32}n_0 = 20$, ${}^{32}n_1 = 4$, ${}^{32}n_2 = 8$ and ${}^{32}T$ is

$${}^{32}T = \begin{bmatrix} 15/20 & 3/4 & 2/8\\ 1/20 & 3/4 & 0/8\\ 4/20 & 1/4 & 3/8 \end{bmatrix}.$$

Replacement ratios are 15:1:4, 3:3:1 and 2:0:3.

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