■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks:... In the context of approximation and interpolation, the general goal is the ANALYTICAL DEFINITION OF A MODEL OF THE CLASSIFICATION PERFORMANCE FUNCTION. During a system's performance evaluation phase – where ground truth information concerning system-classified data is known – an "expert" can assign the desired performance value for a specific classification result tuple. For example, an "expert" assigns a value $p \in [0,1]$ to the tuple (TN, TN, FP, TP, FN); here, $n_3 = 2$, $n_2 = 1$, $n_1 = 1$ and $n_0 = 1$ and $\overline{n}_3 = 2/5$, $\overline{n}_2 = 1/5$, $\overline{n}_1 = 1/5$ and $\overline{n}_0 = 1/5$. This data can be used as an approximation or interpolation constraint for model construction. If one wanted to devise a quadratic model, the performance function would have the form

$$P = p(\overline{n}_0, \ldots, \overline{n}_3) = \sum_{\substack{i+j+k+l \le 2, \\ i,j,k,l \ge 0}} w_{i,j,k,l} \, \overline{n}_0^{\,i} \, \overline{n}_1^{\,j} \, \overline{n}_2^{\,k} \, \overline{n}_3^{\,l} \; .$$

Using "expert-defined" constraits $p_t = p(\overline{n}_0^{\,t}, \overline{n}_1^{\,t}, \overline{n}_2^{\,t}, \overline{n}_3^{\,t})$, $t = 1 \ldots T$, establishes a (usually over-determined) system for the unknown weights (coefficients) $w_{i,j,k,l}$. THE FINAL MODEL $p(\overline{n}_0, \overline{n}_1, \overline{n}_2, \overline{n}_3)$ ALLOWS ONE TO COMPUTE A p-VALUE FOR ANY RESULT TUPLE.

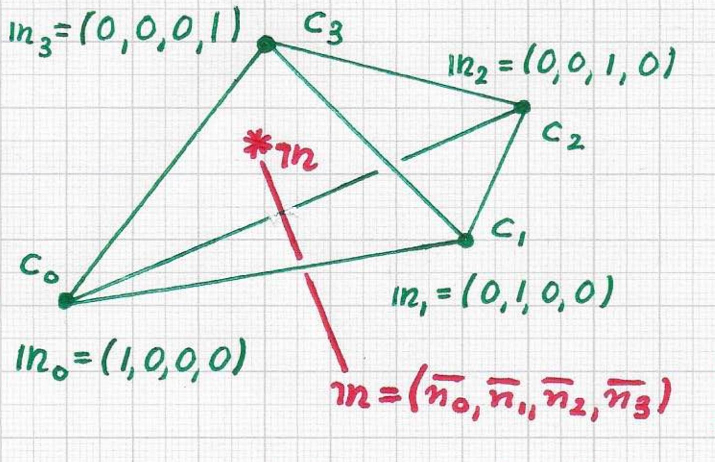
...

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks:... • **Note.** We also must keep in mind the other meaning of "weight":

One also must establish "proper" weight values to the individual, single classification outcomes – TN, TP, FP, FN – and even outcome result **tuples**, e.g., to the 5-tuple (TN, TP, FN, TN, FP). These weights would ultimately define the desired ranking of result tuples, from best to worst. Thus, optimizing a system, by adjusting the system's most important parameter values, must be driven by the objective to get closer and closer to the desired ranking, i.e., the system should generate more and more of the result tuples considered as the best tuples.

One can also view the definition of a 4-variate scalar-valued performance function as follows:



The left figure shows an "embedding tetrahedron" for the binary classification scenario. A point in this tetrahedron is represented in barycentric form, relative to the four corners c_i, $i=0,1,2,3$. A point $m = (\bar{n}_0, \bar{n}_1, \bar{n}_2, \bar{n}_3)$ satisfies $\sum_{i=0}^{3} \bar{n}_i = 1$ and $\bar{n}_i \geq 0$, $i=0,1,2,3$.

...

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

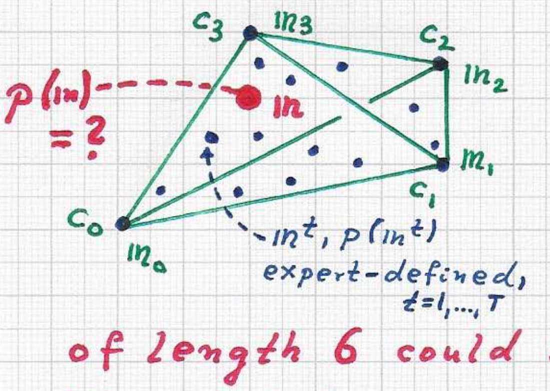
• Laplacian eigenfunctions and neural networks:... An expert can define a finite set of desired performance values p associated with expert-specified quadruples $\mathbb{n} = (\bar{n}_0, \bar{n}_1, \bar{n}_2, \bar{n}_3)$. For example, since $p \in [0,1]$, one could/should assign 1 as p-value to $\mathbb{n} = \mathbb{n}_2 = (0,0,1,0)$ and $\mathbb{n} = \mathbb{n}_3 = (0,0,0,1)$, considering the fact that $\mathbb{n}_2$ represents a classification result outcome of exclusively TP result types, and $\mathbb{n}_3$ represents a classification outcome of exclusively TN result types. (More generally, one should assign 1 as p-value to the edge $(\bar{n}_2 + \bar{n}_3) = 1$, $\bar{n}_2 \geq 0$ and $\bar{n}_3 \geq 0$, since this edge of the "embedding tetrahedron" represents the set of all tuples/points $\mathbb{n}$ corresponding to exclusively TP and TN result tuple types being the normalized component values of $\mathbb{n}$.) As also described on page 16 (1/16/2023), would define p-value interpolation constraints $p(\bar{n}_0^t, \bar{n}_1^t, \bar{n}_2^t, \bar{n}_3^t) = p_t$, $t = 1 \ldots T$, for T quadruples $\mathbb{n}$ where a specific p-value must be the performance value. One must now construct an analytical, continuous performance model function that must satisfy the interpolation conditions, has values between 0 and 1 and has a low degree of / minimal variation.

•••

■ OBJECT AND MATERIALS EIGENFUNCTIONS – Cont'd.

• *Laplacian eigenfunctions and neural networks:...* **Why** is it necessary to define a classification performance function using such a "manual," involved and time-intensive process? The goal is to devise and fine-tune the classification system such that near-optimal performance results — RELATIVE TO A SPECIFIC PERFORMANCE FUNCTION. This goal justifies the process, **since** fine-tuning system parameter values must optimize the chosen performance function.

The "manual" part of performance function specification requires an expert to define performance function values p for expert-selected quadruples $\mathbb{m} = (\bar{n}_0, \bar{n}_1, \bar{n}_2, \bar{n}_3)$. An expert can only specify a finite number of these quadruples and required p-values for these quadruples.



During actual system evaluation, one can choose to use classification result tuples of arbitrary length. For example a result tuple of length 6 could be (TN, FN, TP, FP, FN, TN). The m-tuple is $\mathbb{m} = (2/6, 1/6, 1/6, 2/6)$, and it must be possible to compute $p(\mathbb{m})$ efficiently for this 4-tuple. ...

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks:... Therefore, one must be able to calculate efficiently a p-value for ANY TUPLE $\mathbb{n}$ IN THE CONTINUOUS "BARYCENTRIC DOMAIN" defined by the conditions $\sum \bar{n}_i = 1$ and $\bar{n}_i \geq 0$. Further, the employed interpolation method — interpolating the expert-defined constraints — must generate p-values in the interval $[0,1]$ over the "barycentric domain." The table included here (left) provides examples of mapping classification result tuples of varying length (from 1 to 6) to their corresponding tuple $\mathbb{n} = (\bar{n}_0, \bar{n}_1, \bar{n}_2, \bar{n}_3)$. Such a table would be generated when evaluating a system's classification performance.

classification result tuple	barycentric tuple $\mathbb{n}$
(TN)	(0, 0, 0, 1)
(TN, TP)	(0, 0, 1/2, 1/2)
(TN, TP, FN)	(1/3, 0, 1/3, 1/3)
(TN, TP, FN, FP)	(1/4, 1/4, 1/4, 1/4)
(TN, TP, FN, FP, TN)	(1/5, 1/5, 1/5, 2/5)
(TN, TP, FN, FP, TN, TP)	(1/6, 1/6, 2/6, 2/6)

For each tuple $\mathbb{n}$, a value $p(\mathbb{n})$ is computed during evaluation — WHERE IT IS ESSENTIAL THAT A GLOBAL OR LOCAL INTERPOLATION METHOD IS USED THAT QUICKLY CALCULATES $p(\mathbb{n})$. OVERALL SYSTEM PERFORMANCE WILL BE BASED ON THE SET $\{p(\mathbb{n})\}$ OF ALL p-VALUES OBTAINED, e.g., using the average.

...