■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks:... Considering once again the 8-material example from page 23 (1/29/2023), the resulting normalized relatvive occurrence / counter table for the 16 possible result types (for 4 classes) is the table provided here:

i \ j	0	1	2	3
0	2/8	1/8	0	0
1	1/8	1/8	1/8	0
2	0	0	0	0
3	1/8	0	1/8	0

Normalized relative classification type counter table $\langle \bar{n}_{i,j} \rangle$.

Together with the **weight** table, see p. 22, $\langle w_{i,j} \rangle$, one can compute the overall average performance as

$$p = \frac{1}{40} \cdot \frac{1}{8} \cdot (2 \cdot 40 + 39 + 30 + 40 + 32 + 0 + 2)$$
$$= \frac{1}{320} \cdot 223 \approx 0.697$$

Thus, the general formula for p is

$$p = \sum_{i=0}^{c} \sum_{j=0}^{c} w_{i,j} \cdot \bar{n}_{i,j} \; .$$

• Note. The counter table $\langle \bar{n}_{i,j} \rangle$ of normalized relative classification type occurrences characterizes system performance as a "distribution of classification result types." Once this table has converged, it completely defines the system response, as far as relative result types are concerned.

Therefore, system parameter values must be chosen to optimize the table $\langle \bar{n}_{i,j} \rangle$, subject to ultimately optimizing performance p.

...

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• *Laplacian eigenfunctions and neural networks:...* We now consider the simple 2-class scenario again, where class-0 materials are not dangerous and class-1 materials are dangerous. Viewing even this binary classification as a multi-class classification, the two normalized classification result-type counter / relative occurrence and weight tables are:

$i \backslash j$	0	1
0	$n_{0,0}$	$n_{0,1}$
1	$n_{1,0}$	$n_{1,1}$

$\cdot \frac{1}{L}$,

$i \backslash j$	0	1
0	$W_{0,0}$	$W_{0,1}$
1	$W_{1,0}$	$W_{1,1}$

$\cdot \frac{1}{M}$

Here, L is the total number of result types, and M is the maximal value of the non-negative initial weights $W_{i,j}$, i.e., $M = \max\{W_{i,j}\}_{i,j=0}^{1}$. The initial weights $W_{i,j}$ represent the "degree of danger/cost" assigned to a result type, e.g., $W_{0,0} = W_{1,1} = 0$, $W_{1,0} > W_{0,1} > 0$. It can be assumed that $M = W_{1,0}$. The normalized weights are $\tilde{w}_{i,j} = W_{i,j} / M$, where $M \neq 0$, which are subsequently mapped to the needed "performance weights" $w_{i,j} = 1 - \tilde{w}_{i,j}$, mapping minimal danger to 1 and maximal danger to 0. Denoting the relative occurrences of result types as $\bar{n}_{i,j} = n_{i,j} / L$, $L > 0$, the average system performance is $p = \sum_{i,j=0}^{1} w_{i,j} \cdot \bar{n}_{i,j}$.

...

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks:... We consider an example based on ten generated result types:

$$\langle n_{i,j}\rangle = \begin{bmatrix} 6 & 1 \\ 1 & 2 \end{bmatrix} \Rightarrow \langle \bar{n}_{i,j}\rangle = \begin{bmatrix} 6/10 & 1/10 \\ 1/10 & 2/10 \end{bmatrix} ; \langle W_{i,j}\rangle = \begin{bmatrix} 0 & 1 \\ 4 & 0 \end{bmatrix}$$

$$\Rightarrow \langle \tilde{w}_{i,j}\rangle = \begin{bmatrix} 0 & 1/4 \\ 1 & 0 \end{bmatrix} \Rightarrow \langle w_{i,j}\rangle = \begin{bmatrix} 1 & 3/4 \\ 0 & 1 \end{bmatrix}$$

$\Rightarrow$ average performance $p = 6/10 + 3/40 + 2/10 = 7/8$.

• THE IDEAL, PERFECT CLASSIFICATION SYSTEM SATISFIES THE EQUATIONS

(i) $n_{0,0} + n_{1,1} = L$ and (ii) $n_{0,1} = n_{1,0} = 0$.

THUS, THE AVERAGE SYSTEM PERFORMANCE IS $p = 1$, USING WEIGHTS $w_{0,0} = w_{1,1} = 1$.

• Considering the values used in the example, one obtains the performance formula

$$\begin{aligned} p &= \bar{n}_{0,0}\cdot 1 + \bar{n}_{0,1}\cdot 3/4 + \bar{n}_{1,0}\cdot 0 + \bar{n}_{1,1}\cdot 1 \\ &= 1 - (\bar{n}_{0,1} + \bar{n}_{1,0} + \bar{n}_{1,1}) + 3/4\,\bar{n}_{0,1} + \bar{n}_{1,1} \\ &= 1 - 1/4\,\bar{n}_{0,1} - \bar{n}_{1,0} = 1 - (\bar{n}_{0,1} + 4\bar{n}_{1,0})/4 . \end{aligned}$$

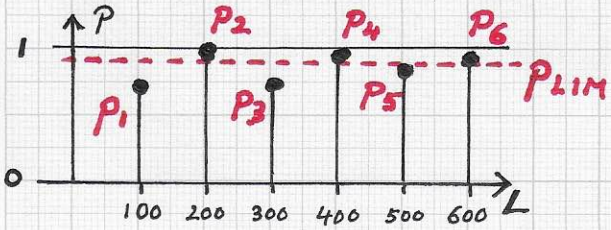
• The value of p should – generally – converge to a "limit performance value," i.e.,

$$p_{LIM} = \lim_{L\to\infty} \left(1 - (\bar{n}_{0,1} + 4\bar{n}_{1,0})/4\right) .$$

...

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

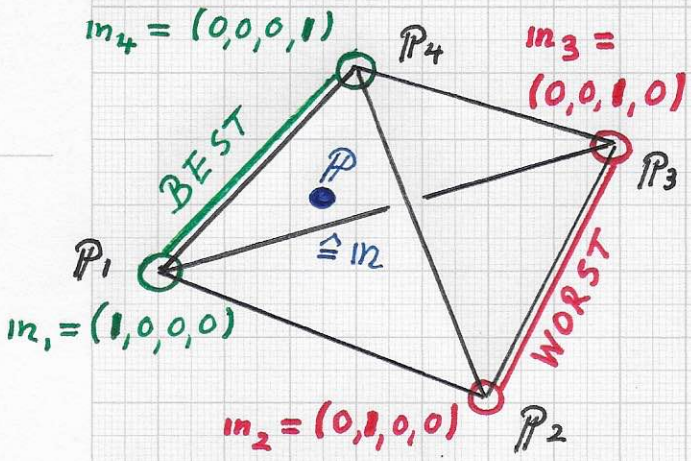
• *Laplacian eigenfunctions and neural networks:...* Generally, it can be assumed that this average value of classification system performance indeed converges to a limit value for $L \to \infty$, assuming that the system always "acts in the same way" (without ever changing its parameter value settings) and the ratio of non-dangerous (class-0) and dangerous (class-1) materials always remains "unchanged." The figure (left) sketches a possible set of six p-values obtained for a linearly increasing number of materials to be classified. **One can employ extrapolation methods generating an estimate of the p_{LIM}-value from experimentally computed values $p_1, p_2, p_3, \ldots$** Alternatively – instead of increasing the number of materials (L) – one can keep the number L fixed, e.g., L = 100, and generate a large set of p-values for different material streams, each stream being of length L. In this case, p-values would "cluster," and the cluster center value should converge to p_{LIM} as well. **If a system had to satisfy the requirement $p_{LIM} > p_{min}$, where p_{min} is the minimal performance limit, one would have to demonstrate that this average performance is observed.** ...



■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks:...

Of course, one must assume that it is practically necessary to satisfy more stringent requirements. For example, it might be required that the described average performance value satisfies $p_{LIM} > p_{min}$ AND that $\bar{n}_{0,1} < \bar{n}_{0,1}^{max}$ AND that $\bar{n}_{1,0} < \bar{n}_{1,0}^{max}$.



Correspondence between points and in-tuples, using 4 coordinates.

The tuples $in = (\bar{n}_{0,0}, \bar{n}_{0,1}, \bar{n}_{1,0}, \bar{n}_{1,1})$ can be visualized via a "barycentric correspondence" to points in the interior or on the boundary of a tetrahedron – by representing points with four barycentric coordinates relative to the four corners of the tetrahedron.

The figure shown here illustrates this "barycentric correspondence": We can associate the in-tuples $in_1 = (1,0,0,0)$, $in_2 = (0,1,0,0)$, $in_3 = (0,0,1,0)$ and $in_4 = (0,0,0,1)$ with the tetrahedron's corner points p_1, p_2, p_3 and p_4, respectively. Given an in-tuple $in = (\bar{n}_{0,0}, \bar{n}_{0,1}, \bar{n}_{1,0}, \bar{n}_{1,1})$, we can define its corresponding point as the point $p = \bar{n}_{0,0}\, p_1 + \bar{n}_{0,1}\, p_2 + \bar{n}_{1,0}\, p_3 + \bar{n}_{1,1}\, p_4$. As indicated in the figure, BEST performance corresponds to the edge $\overline{p_1 p_4}$, while WORST performance corresponds to the edge $\overline{p_2 p_3}$.

...