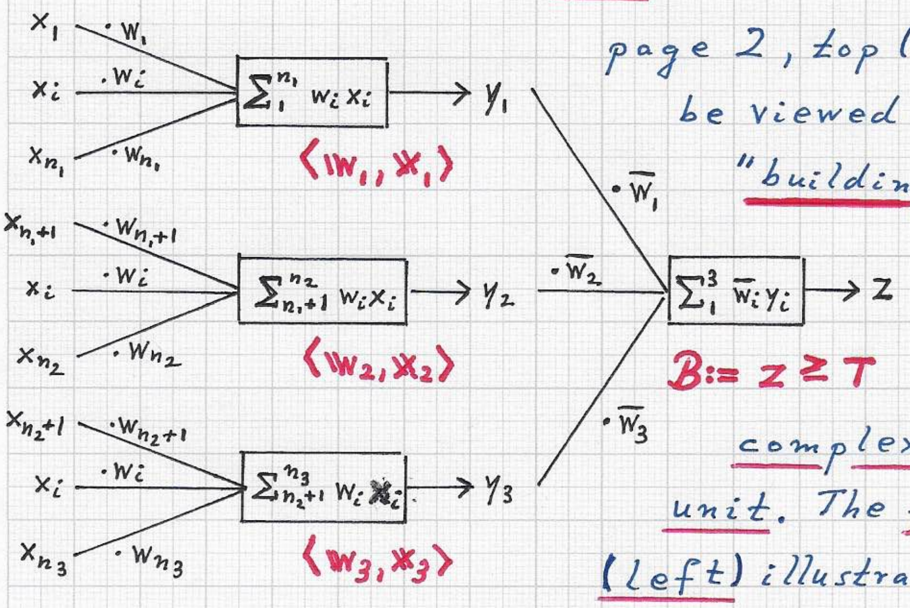
OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks:... The simple threshold logic unit shown in the figure on page 2, top (2/16/2023) can be viewed as a basic "building block" serving merely as one component of a much more complex threshold logic unit. The figure shown here (left) illustrates a possibility to break down an input vector $(x_1,\dots,x_{n_1},x_{n_1+1},\dots,x_{n_2},x_{n_2+1},\dots,x_{n_3})^T$ into three input vectors $(x_1,\dots,x_{n_1})^T$, $(x_{n_1+1},\dots,x_{n_2})^T$ and $(x_{n_2+1},\dots,x_{n_3})^T$ with associated weight vectors $(w_1,\dots,w_{n_1})^T$, $(w_{n_1+1},\dots,w_{n_2})^T$ and $(w_{n_2+1},\dots,w_{n_3})^T$, respectively. For example, one can call these vectors $\mathbb{x}_1$, $\mathbb{x}_2$ and $\mathbb{x}_3$ and $\mathbb{w}_1$, $\mathbb{w}_2$ and $\mathbb{w}_3$, respectively, as indicated in the figure.



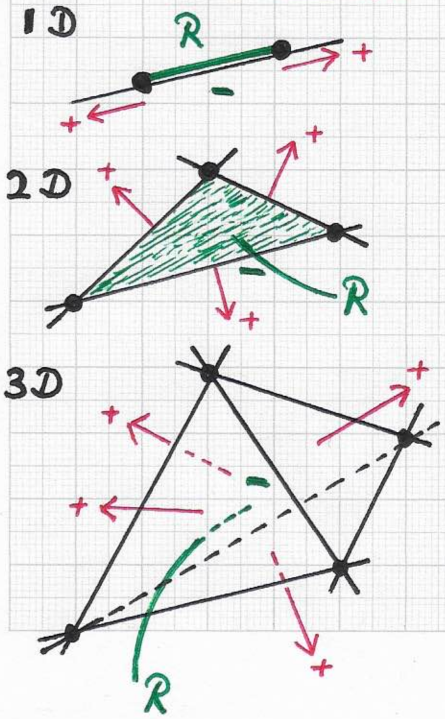
Breaking down the input in such a way should make "semantic sense" and, ideally, support a better understanding of data processing, analysis and classification results. Here, we first produce the intermediate processing results $y_1 = \langle \mathbb{w}_1, \mathbb{x}_1 \rangle$, $y_2 = \langle \mathbb{w}_2, \mathbb{x}_2 \rangle$ and $y_3 = \langle \mathbb{w}_3, \mathbb{x}_3 \rangle$. These three (real-valued) outputs generate the final output $z = \langle \bar{\mathbb{w}}, \mathbb{y} \rangle$. Threshold logic defines the value of $\mathbf{B}$ as $z \geq T$.

...

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks: ... This view of a FORWARD computational neural network (last page) is a rather simplified view. Generally, the edge connectivity between "processing nodes" can be much more complex. Further, "processing nodes" can generate output that includes possibly **multiple** real, scalar-valued and real, vector-valued quantities – and also the value(s) of a Boolean variable(s) calculated via threshold logic. We do not consider these more general network (graph) topologies and processing capabilities at this point.

! We will now discuss in more detail the relationship between a "half-space test" and logic-based decision-making for data classification. A fundamental concept is the definition of a convex region R as the intersection of half-spaces. This concept is illustrated in the left figure for the 1D, 2D and 3D cases. In the 1D case, R is the intersection of two negative '–' half-spaces defined by two points '•' and their associated positive '+' (outward-pointing) normals '→'.



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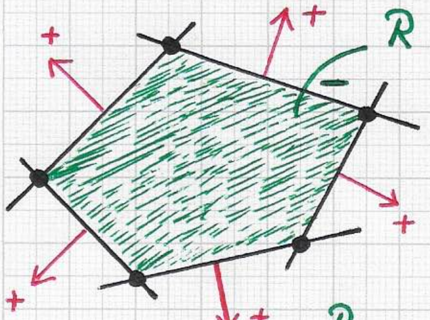
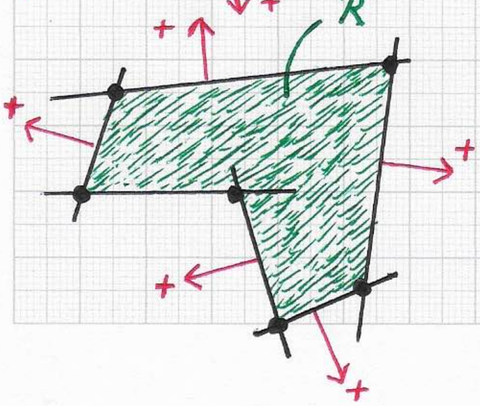
OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks:... In the 2D (3D) case, R is the intersection of three (four) negative '–' half-spaces defined by three (four) edges (faces), with corner points '•', and their associated positive '+' (outward-pointing) normals '⟶'. (These examples show the simplest convex regions in 1D, 2D and 3D space, i.e., the simplices "line segment," "triangle" and "tetrahedron," respectively.) In the context of data classification, we must understand the dimension of the space as dimension of the data's feature space, and the region R as the region representing the set of all possible feature points/vectors defining materials belonging to the same material class.

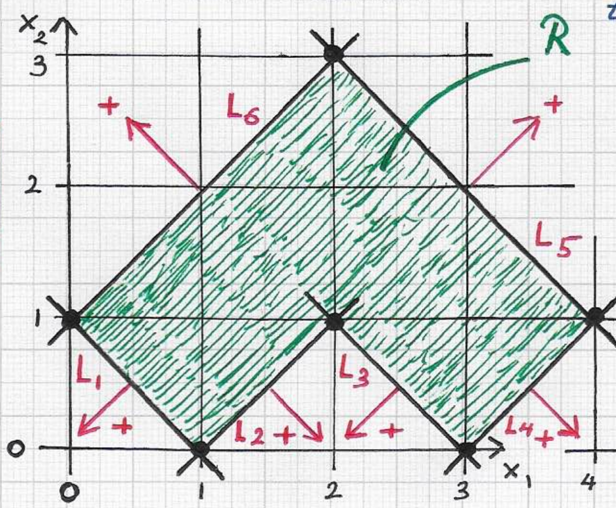




The two examples shown in the left figure illustrate that the convex region R (top) can be defined as the intersection of the indicated five negative '–' half-spaces, while the non-convex region R (bottom) cannot be obtained as the intersection of the indicated six negative '–' half-spaces. The non-convex region can be defined as union of convex sub-regions. ...

OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

- Laplacian eigenfunctions and neural networks:...



The example illustrated in the left figure shows a non-convex region R in the plane to be defined as the union of two convex sub-regions, each represented as the intersection of four negative half-spaces. The six lines (linear polynomials) involved in the definition of R are called L_i, $i=1\ldots6$. The six (oriented) lines and their implicit equations are:

L_1: $-x_1-x_2+1=0$, L_2: $x_1-x_2-1=0$,
L_3: $-x_1-x_2+3=0$, L_4: $x_1-x_2-3=0$,
L_5: $x_1+x_2-5=0$, L_6: $-x_1+x_2-1=0$.

One can now view L_i as a bivariate linear polynomial that defines the respective oriented line, i.e., $L_1(x_1,x_2)=-x_1-x_2+1$, $L_2(x_1,x_2)=x_1-x_2-1$, ..., $L_6(x_1,x_2)=-x_1+x_2-1$. We define the first convex sub-region as the set of points satisfying

$$L_1(x_1,x_2)\le 0 \wedge L_2(x_1,x_2)\le 0 \wedge L_5(x_1,x_2)\le 0 \wedge L_6(x_1,x_2)\le 0$$

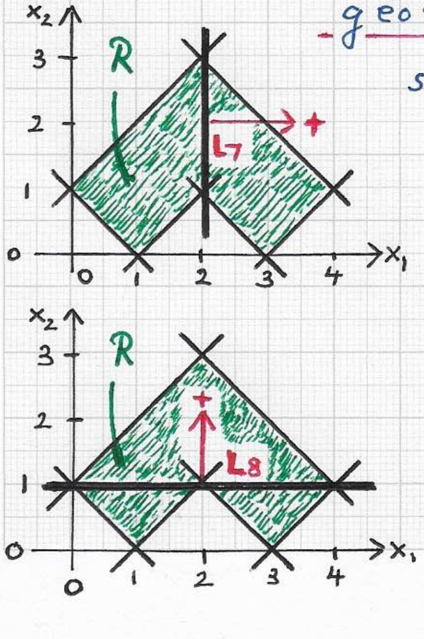
The second convex sub-region is the set of points satisfying the Boolean condition

$$L_3(x_1,x_2)\le 0 \wedge L_4(x_1,x_2)\le 0 \wedge L_5(x_1,x_2)\le 0 \wedge L_6(x_1,x_2)\le 0 .$$

Points (x_1,x_2) satisfying one or both of these conditions belong to R....

OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks:... In order to simplify and make more compact the notation, we can, for example, simply write "L_i" as a simplification for the expression "$L_i(x_1, x_2) \le 0$". Thus, we can understand L_i as a Boolean variable, with value TRUE (1) or FALSE (0). Further, we can also use another simplification: We write the logical AND operator '$\wedge$' in an expression like "$L_i \wedge L_j$" like a multiplication, i.e., "$L_i \wedge L_j$" = "$L_i \cdot L_j$" = "$L_i L_j$". By using these conventions to define the set of points $(x_1, x_2)^T$ inside or on the boundary of the non-convex region R defined on the previous page, this set is the set of points for which the value of the Boolean expression $L_1 L_2 L_5 L_6 \vee L_3 L_4 L_5 L_6$ is TRUE. Of course, many allowable combinatorial/geometrical possibilities exist to represent R as the union of convex sub-regions.



The two sketches shown here (left) include two more implicitly defined lines:

L_7: $x_1 - 2 = 0$, L_8: $x_2 - 1 = 0$.

With these lines one can write R as

$R = \{(x_1, x_2)^T \mid L_1 L_2 L_6 L_7 \vee L_3 L_4 L_5 \overline{L_7}\}$ or

$R = \{(x_1, x_2)^T \mid L_1 L_2 L_8 \vee L_3 L_4 L_8 \vee L_5 L_6 \overline{L_8}\}$.

Here, we use the notation "$\overline{L_i}$" to represent "$\neg L_i$".

...