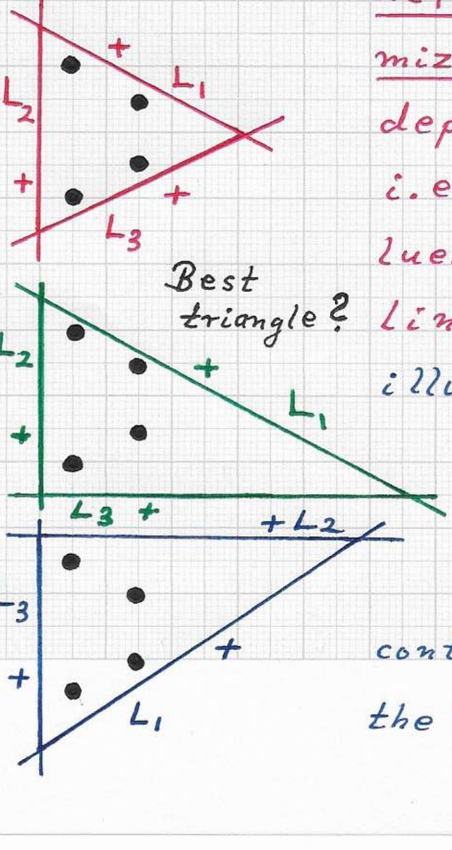
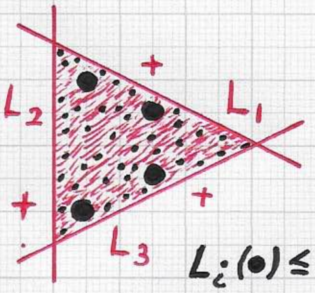
■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks:... In summary, one sees that a "point matrix" X defines the half-space conditions for the unknown vector $\mathbb{w}$ containing the parameters of all linear equations L_i. The resulting system is of the form $X \cdot \mathbb{w} \leq \mathbb{1}$. The geometrical meaning of this system of 12 inequalities for the determination of three lines is straightforward: Given the 4 points $\mathbb{x}_k$ in the plane (not being colinear), find (the best!) L_1, L_2 and L_3 such that the points $\mathbb{x}_k$ are all in the negative half-spaces defined by the lines. Of course an infinite number of lines exist that satisfy this condition. Thus, it is necessary to define a (cost) function to be optimized — where the function's value depends on the chosen three lines, i.e., the chosen nine parameter values defining the three implicit line equations L_i. The left figure illustrates this specific case: Three possibilities are shown for four points where the lines define the edges of a triangle that contains the points in its interior — the intersection of the lines' negative half-spaces. ...



OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks:… The figure shown here (left, top) merely is included to support



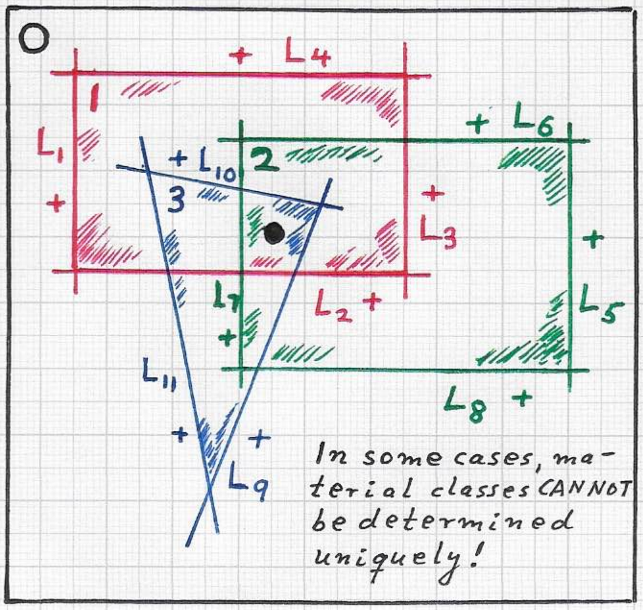
one's intuition that this particular set of lines L_i "constructed for the four **bold** points '●' " is a very good

$L_i(\bullet) \leq 0,$

set – by considering the illustrated

$L_i(\cdot) \leq 0$

fact that also all other points '·' lie inside the triangle defined by the three negative half-spaces. Thus, the goal is to construct the set of Lines L_i (number of lines and parameter values of lines) such that the overall performance function of the classification system is optimized. The



Left figure shows a case where two rectangles and one triangle intersect; they are "contained" in the large rectangle '0'. The sketch assumes that the regions called '1', '2' and '3' are the regions associated with material classes 1, 2 and 3, respectively. The regions '1', '2' and '3' are understood as **CONTINUA**, representing the space where all possible points ✻ of classes 1, 2 or 3 must reside. Considering the point '●', it is not possible to determine its specific class. •••

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks:... We can use the notation from p. 10 (2/22/2023) to establish the eight possible classification outcomes of a point $\times$ inside the large rectangle '0' – using logical Boolean expressions based on the lines L_i, $i = 1 \dots 11$, and the associated half-spaces. The eight possibilities are:

Class 0 ——— $\overline{L_1 L_2 L_3 L_4} \wedge \overline{L_5 L_6 L_7 L_8} \wedge \overline{L_9 L_{10} L_{11}}$

Class 1 ——— $L_1 L_2 L_3 L_4 \wedge \overline{L_5 L_6 L_7 L_8} \wedge \overline{L_9 L_{10} L_{11}}$

Class 2 ——— $\overline{L_1 L_2 L_3 L_4} \wedge L_5 L_6 L_7 L_8 \wedge \overline{L_9 L_{10} L_{11}}$

Class 3 ——— $\overline{L_1 L_2 L_3 L_4} \wedge \overline{L_5 L_6 L_7 L_8} \wedge L_9 L_{10} L_{11}$

Class 1 or class 2 ——— $L_1 L_2 L_3 L_4 \wedge L_5 L_6 L_7 L_8 \wedge \overline{L_9 L_{10} L_{11}}$

Class 1 or class 3 ——— $L_1 L_2 L_3 L_4 \wedge \overline{L_5 L_6 L_7 L_8} \wedge L_9 L_{10} L_{11}$

Class 2 or class 3 ——— $\overline{L_1 L_2 L_3 L_4} \wedge L_5 L_6 L_7 L_8 \wedge L_9 L_{10} L_{11}$

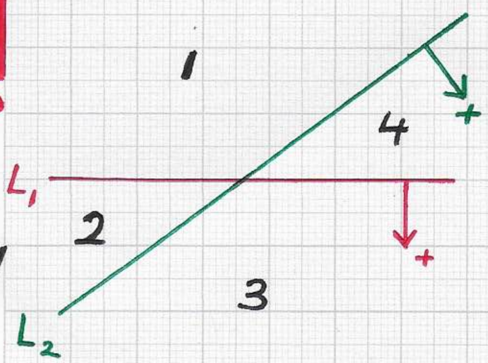
Class 1 or class 2 or class 3 — $L_1 L_2 L_3 L_4 \wedge L_5 L_6 L_7 L_8 \wedge L_9 L_{10} L_{11}$

Here, the NOT ($\neg$) operator is written as '$\overline{}$'.
Eight cases arise as a consequence of $\sum_{j=0}^{3} \binom{3}{j} = 2^3 = 8$.

Again, the important point is the fact that algebra (half-space equations), geometry (regions in space), logic (Boolean expressions) and data classification are intimately related – and, in several contexts, are equivalent means for defining and solving a problem. One of the main reasons for presenting the example above involving four classes (0, 1, 2, 3) is the fundamental truth that classes might not be separable; it might not be possible to determine a material's class uniquely. ...

OBJECT AND MATERIAL EIGENFUNCTIONS — Cont'd.

• Laplacian eigenfunctions and neural networks:...



We consider another example to demonstrate the relationship between geometry, algebra and logic, illustrated in the left figure. The two intersecting lines L_1 and L_2 define four sub-regions in the plane, called **1, 2, 3 and 4**.

(L_i is simultaneously understood and used as notation for the associated oriented line and the Boolean logic variable with its truth value for a point $\underline{x}$ assigned as $L_i := (L_i(\underline{x}) \leq 0)$ — where $L_i(\underline{x})$ on the right-hand side is the line's linear polynomial evaluated for the arbitrary point $\underline{x}$.) The two lines sketched in the figure above define $16 = 2^4 = 4^2$ possibilities to represent regions/sub-spaces via Boolean expressions using L_i-variables:

R	B	R	B
$\emptyset$	F	$2 \cup 3$	$\overline{L}_1$
1	$L_1 L_2$	$1 \cup 3$	$L_1 L_2 \vee \overline{L}_1 \overline{L}_2$
2	$\overline{L}_1 L_2$	$2 \cup 4$	$\overline{L}_1 L_2 \vee L_1 \overline{L}_2$
3	$\overline{L}_1 \overline{L}_2$	$1 \cup 2 \cup 3$	$L_2 \vee \overline{L}_1 \overline{L}_2$
4	$L_1 \overline{L}_2$	$1 \cup 2 \cup 4$	$L_2 \vee L_1 \overline{L}_2$
$1 \cup 2$	L_2	$1 \cup 3 \cup 4$	$L_1 \vee \overline{L}_1 \overline{L}_2$
$3 \cup 4$	$\overline{L}_2$	$2 \cup 3 \cup 4$	$\overline{L}_1 \vee L_1 \overline{L}_2$
$1 \cup 4$	L_1	$1 \cup 2 \cup 3 \cup 4$	T

The spatial regions (R) are defined by the associated Boolean expressions (B). Considering the laws/rules applicable to Boolean algebra, listed expressions B could be replaced by equivalent ones.

An expression B is optimal when it can be comupted most efficiently.

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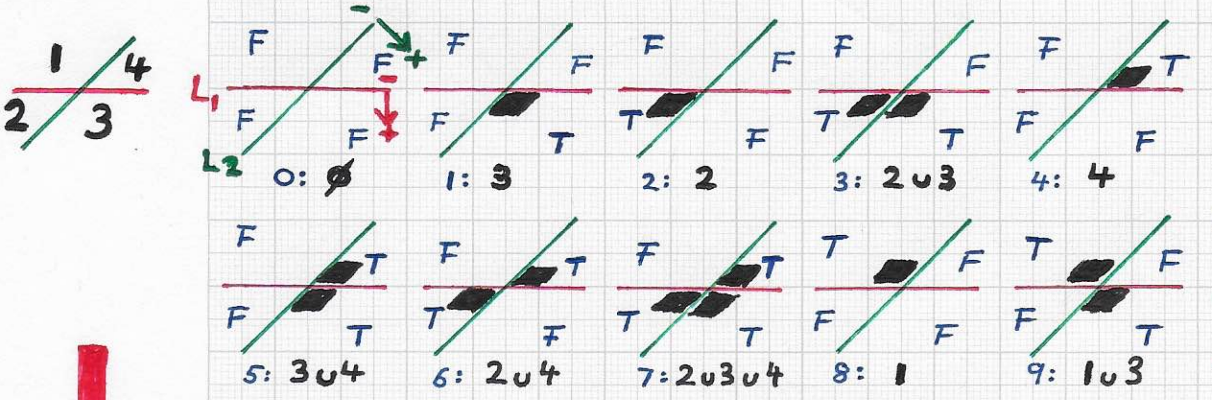
■ OBJECT AND MATERIAL EIGENFUNCTIONS - Cont'd.

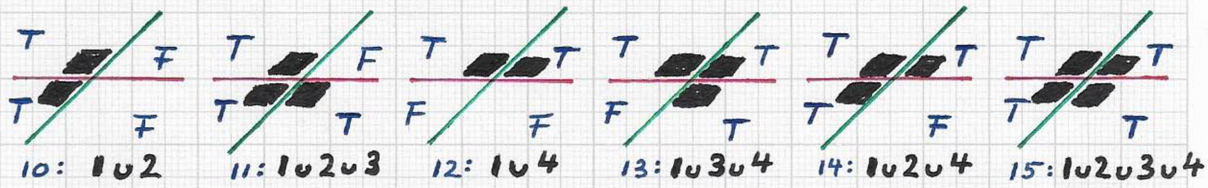
• Laplacian eigenfunctions and neural networks:...

	b_1	T	T	F	F	function
	b_2	T	F	T	F	
	0	F	F	F	F	F
	1	F	F	F	T	NOR
*	2	F	F	T	F	$b_1 \nleftarrow b_2$
	3	F	F	T	T	$\neg b_1$
*	4	F	T	F	F	$b_1 \nrightarrow b_2$
	5	F	T	F	T	$\neg b_2$
	6	F	T	T	F	XOR
	7	F	T	T	T	NAND
	8	T	F	F	F	AND
	9	T	F	F	T	XNOR
	10	T	F	T	F	b_2
*	11	T	F	T	T	$b_1 \rightarrow b_2$
	12	T	T	F	F	b_1
*	13	T	T	F	T	$b_1 \leftarrow b_2$
	14	T	T	T	F	OR
	15	T	T	T	T	T

*
2 converse non-implication
4 material non-implication
11 material implication
13 converse implication

• **Note.** Considering two Boolean variables b_1 and b_2, $16 = 4^2$ Boolean functions can be defined by the possible quadruples with T or F component values. The left table shows these 16 functions and provides typically used names for these functions. Thus, it is also possible to relate the 16 region/sub-space possibilities summarized in the table from the previous page and the 16 Boolean functions. The 16 Boolean functions 0, 1, ..., 15 listed in the table are illustrated in the left sketches.





These sketches show the 16 possible ways to combine the four regions 1, 2, 3 and 4.

...