Bernd Hamann 3/9/2023

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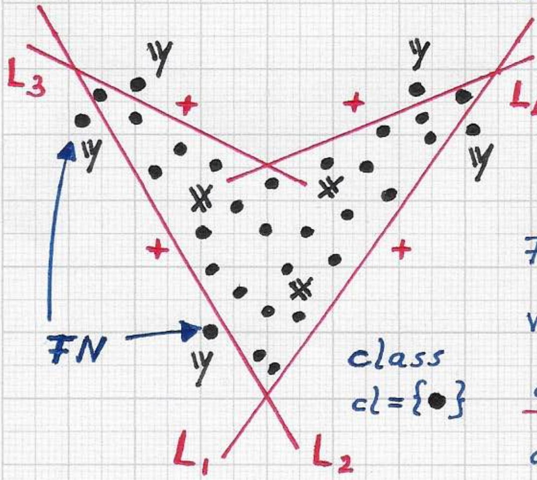
OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks:...



The left figure illustrates a non-convex configuration that could potentially define the predicate of a point x as

$$P(x) = L_1 L_2 L_3 \vee L_1 L_2 L_4 .$$

For all points y, $P(y) = FALSE$, which is an incorrect classification of these points, as it is assumed that all shown points x and y are class-cl points.

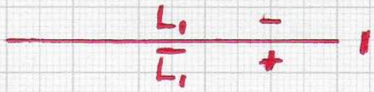
Nevertheless, it is assumed here that this specific placement of four lines L_i is optimal — optimal in the sense that it maximizes (minimizes) the numbers of TN and TP (FN and FP) classifications. "When limited to four lines, this set of four lines is one set that allows one to formulate a Boolean expression of a predicate P that optimizes classification performance." Therefore, the goal is to find a Boolean expression — based on the placed lines L_i — that allows one to optimally approximate the class-cl spatial region(s) via a "minimal combination of Boolean variables L_i" — leading to a best-possible predicate definition for class cl. The following important question arises: Given lines L_i, $i=1...r$, how many regions and equivalent Boolean expressions can be constructed?

...

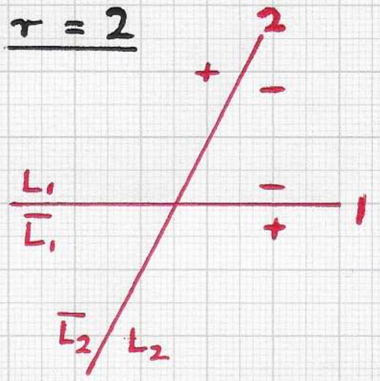
■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks:...

• $\tau = 1$



• $\tau = 2$



$$\boxed{\sum_{j=1}^{2^\tau} \binom{2^\tau}{j} = 2^{2^\tau} = \begin{bmatrix} 4, & \tau = 1 \\ 16, & \tau = 2 \\ \ldots & \end{bmatrix}}$$

In the left figures, we merely show the indices **1** and **2** next to the respective lines that define positive **(+)** and negative **(–)** half-spaces. The Boolean variables L_i and $\overline{L_i}$ are represented next to the lines as well. Again, $\overline{L_i} := \neg L_i$. The goal is to determine how all possible regions in the plane are related to corresponding Boolean expressions involving the Boolean variables L_i, $i = 1 \ldots \tau$. It is possible to define a specific region in space via multiple equivalent expressions; an expression can be called optimal when its computational evaluation for a point x is maximally efficient. Therefore, one should first establish the list of all combinatorially possible region definitions, given τ lines, and subsequently determine region-defining Boolean expressions. One line **1** induces 2 definitions $(-, +)$; two lines **1** and **2** induce 4 definitions $(--, -+, +-, ++)$; three lines **1**, **2** and **3** induce 8 definitions $(---, --+, \ldots, +++)$; and lines **1**, ... and τ induce 2^τ definitions.

...

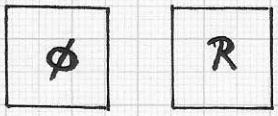
■ OBJECT AND MATERIAL EIGENFUNCTIONS — Cont'd.

• Laplacian eigenfunctions and neural networks:... Thus, the 2^r induced formal definitions allow one to choose subsets of cardinality $0, 1, 2, \ldots, 2^r-1, 2^r$ — leading to the list of all combinatorially possible final region definitions. When ignoring the actual geometrical placement of the lines and especially degenerate, singular line configurations, the TOTAL NUMBER OF POSSIBLE REGIONS THAT CAN BE ESTABLISHED FORMALLY IS 2^{2^r}, see formula on previous page. The table (left) shows how rapidly this number increases.

r	2^{2^r}
0	2
1	4
2	16
3	256
4	65536
5	4294967296

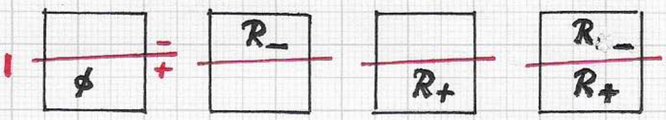
We provide sketches of the relevant geometry for low r values:

• $r = 0$



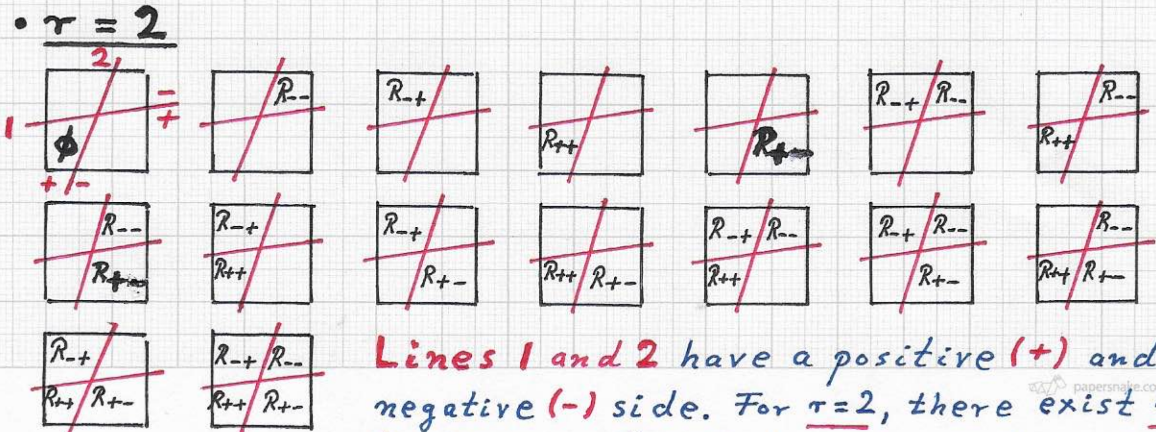
ϕ empty set
R entire space

• $r = 1$



$R = R_- \cup R_+$

• $r = 2$



$R = R_{--} \cup R_{-+} \cup R_{++} \cup R_{+-}$

Lines 1 and 2 have a positive (+) and negative (−) side. For $r=2$, there exist 4 (sub-)regions $R_{..}$ that can be combined in $1+4+6+4+1 = 16$ ways. ...

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3/11/2023

■ OBJECT AND MATERIAL EIGENFUNCTIONS - Cont'd.

- Laplacian eigenfunctions and neural networks:... For the computation of the predicate $P(x)$ of a point, we must evaluate a/the Boolean expression(s) that logically define(s) the region(s) that must be considered for classification decision-making. We summarize Boolean expressions for the sketched cases with r values of 0, 1 and 2. It is relevant to note that the laws/rules and theorems of Boolean algebra make possible the definition of these regions via different yet equivalent Boolean expressions. We provide a few of them.

r	region	predicate $P(x)$
0	$\emptyset$	F (FALSE)
	R	T (TRUE)
1	$\emptyset$	F
	R_-	L_1
	R_+	$\overline{L}_1$
	R	T
2	$\emptyset$	F
	R_{--}	$L_1 L_2$
	R_{-+}	$L_1 \overline{L}_2$
	R_{++}	$\overline{L}_1 \overline{L}_2$
	R_{+-}	$\overline{L}_1 L_2$
	$R_{--} \cup R_{-+}$	$L_1 L_2 \vee L_1 \overline{L}_2 = L_1 (L_2 \vee \overline{L}_2) = L_1$

region	predicate $P(x)$
$R_{--} \cup R_{++}$	$L_1 L_2 \vee \overline{L}_1 \overline{L}_2$
$R_{--} \cup R_{+-}$	$L_1 L_2 \vee \overline{L}_1 L_2 = L_2 (L_1 \vee \overline{L}_1) = L_2$
$R_{-+} \cup R_{++}$	$= \ldots = \overline{L}_2$
$R_{-+} \cup R_{+-}$	$L_1 \overline{L}_2 \vee \overline{L}_1 L_2$
$R_{++} \cup R_{+-}$	$= \ldots = \overline{L}_1$
$R_{--} \cup R_{-+} \cup R_{++}$	$L_1 L_2 \vee L_1 \overline{L}_2 \vee \overline{L}_1 \overline{L}_2 = L_1 \vee \overline{L}_1 \overline{L}_2 = L_1 \vee \overline{(L_1 \vee L_2)}$
$R_{--} \cup R_{-+} \cup R_{+-}$	$= \ldots = L_1 \vee \overline{L}_1 L_2$
$R_{--} \cup R_{++} \cup R_{+-}$	$= \ldots = L_2 \vee \overline{(L_1 \vee L_2)}$
$R_{-+} \cup R_{++} \cup R_{+-}$	$= \ldots = \overline{L}_1 \vee L_1 \overline{L}_2$
R	T

Complete table of all possible regions with predicates $P(x)$.

...

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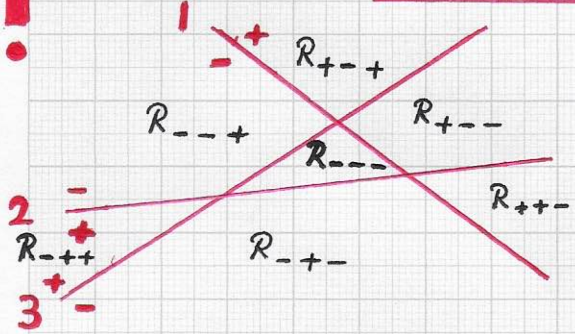
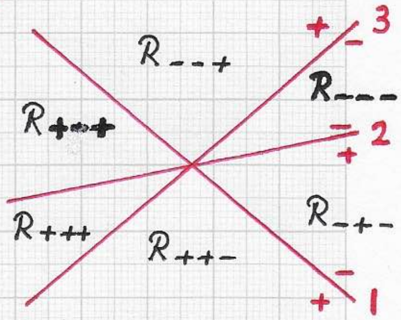
■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

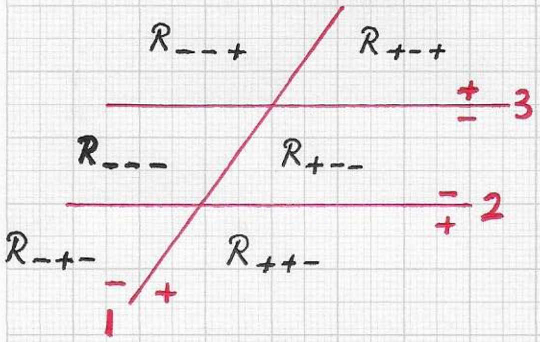
- Laplacian eigenfunctions and neural networks: ... We consider the case $r=3$ next. "Formally" one can define the possible regions combinatorially, leading to the number of $\sum_{j=1}^{2^3} \binom{2^3}{j} = 2^{2^3} = 256$ possibilities.



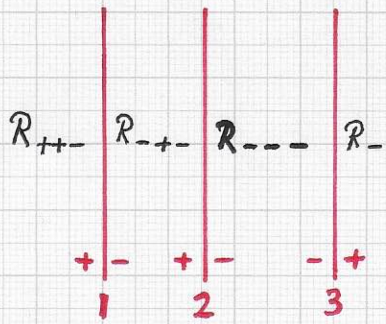
The figures shown on this page clarify that the formally possible 256 cases are geometrically not viable in the plane.



Three lines intersecting in point-degenerate.



Only the example sketched in the top-left figure includes a (sub-)region, i.e., R_{---}, that has a finite area. Of course, one would generally "enclose" all these open geometrical configurations with a bounding (hyper-)box, (hyper-)sphere or other primitive.



Top: Lines in "general position" – not degenerate; middle: Two lines parallel; bottom: Three lines parallel.

The main purpose of this discussion of definining "classification decision boundaries and regions" via simple "line/linear building blocks" is the recognition of combinatorial possibilities. ...