■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks:...

8) The calculation of the multi-resolution approximation of an analytically unknown "class region" **R** is complex and **COMPUTATIONALLY EXPENSIVE.** This fact is irrelevant for the use and application of the multi-resolution structure of **R**: The multi-resolution structure computation, via a large set of given classified (ground truth) sample points ⁎ belonging to the class of interest, is done in a pre-processing phase; it is relevant for the classification of an unclassified point ⁎ that the given multi-resolution structure defines sequences of Boolean expressions that lead to a highly efficient classification computation. The two sequences of approximations of R — one exterior sequence ${}^0R, {}^1R, {}^2R, \ldots$ and one interior sequence ${}^0r, {}^1r, {}^2r, \ldots$ — are designed for highly efficient classification of a point ⁎. Initially, The regions ${}^0R, {}^1R, {}^2R, \ldots$ are "shrinking supersets" of R, and the regions ${}^0r, {}^1r, {}^2r, \ldots$ are "expanding subsets" of R. The regions ${}^0R$ and ${}^0r$ are the simplest super-set **and** subset regions of R, i.e., their definitions via half-spaces involve only small numbers of hyper-planes and Boolean operators. **The regions ${}^jR$ and ${}^jr$**, and their definitions, become increasingly complex with increasing index j. ...

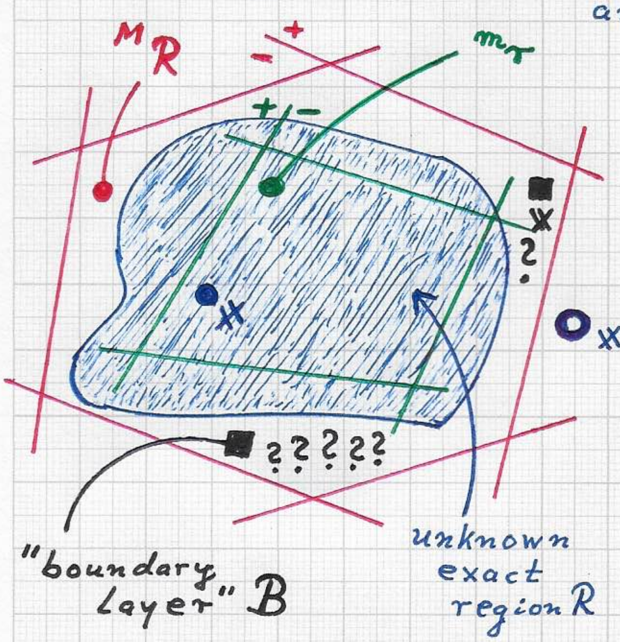
$$
\begin{array}{c}
{}^0R \\ \cup \\ {}^1R \\ \cup \\ {}^2R \\ \cup \\ \vdots \\ R \\ \vdots \\ \cup \\ {}^2r \\ \cup \\ {}^1r \\ \cup \\ {}^0r
\end{array}
$$

! !

!

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks:...



9) Two regions are special. They are sketched in the left figure. They are called ${}^{M}R$ and ${}^{m}r$. The region ${}^{M}R$ is the last region in the exterior sequence ${}^{0}R, {}^{1}R, {}^{2}R, \ldots, {}^{M}R$; the region ${}^{m}r$ is the last region in the interior sequence ${}^{0}r, {}^{1}r, {}^{2}r, \ldots, {}^{m}r$. We assume here that the following statements are true:

i) Given a point $*$ to be classified,

IF ($* \in {}^{0}r \vee * \in {}^{1}r \vee \ldots \vee * \in {}^{m}r$)

THEN ($*$ belongs to class).

ii) Given a point $*$ to be classified,

IF ($* \notin {}^{0}R \vee * \notin {}^{1}R \vee \ldots \vee * \notin {}^{M}R$)

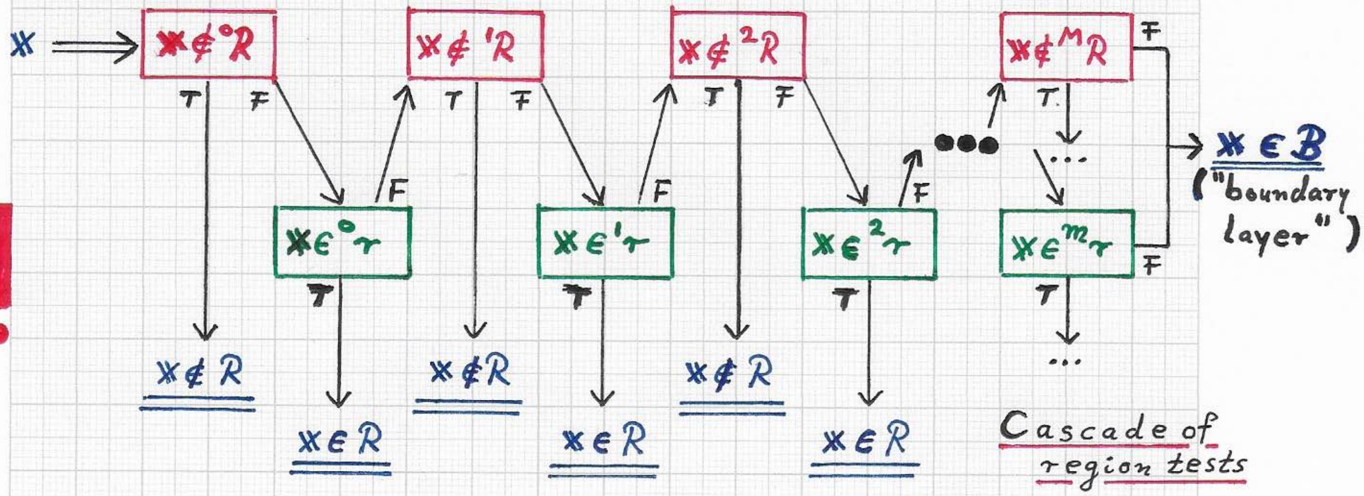
THEN ($*$ does not belong to class).

Further, the "boundary layer" is the region (set) defined as $({}^{M}R - {}^{m}r)$. This "boundary layer" region can contain points $* = ■$ that belong or not belong to the class being considered. This region could be called "uncertainty region" or "undecidability region". BY CONSIDERING A SET OF POINTS $*$, WHERE EACH $* \in ({}^{M}R - {}^{m}r)$ AND ONE KNOWS WHETHER A POINT $*$ BELONGS TO THE CLASS OR NOT, ONE CAN ESTIMATE THE FRACTION OF IN-CLASS POINTS.

...

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks:... **10)** The sets of increasingly more complicated and more precise approximations of the "class region R," i.e., the sets $\{{}^0R, {}^1R, \ldots, {}^MR\}$ and $\{{}^0r, {}^1r, \ldots, {}^mr\}$, make it possible to design and use an "alternating cascading procedure" for classification decision-making." This procedure can be viewed as a specific type of decision tree. It can be sketched like this:



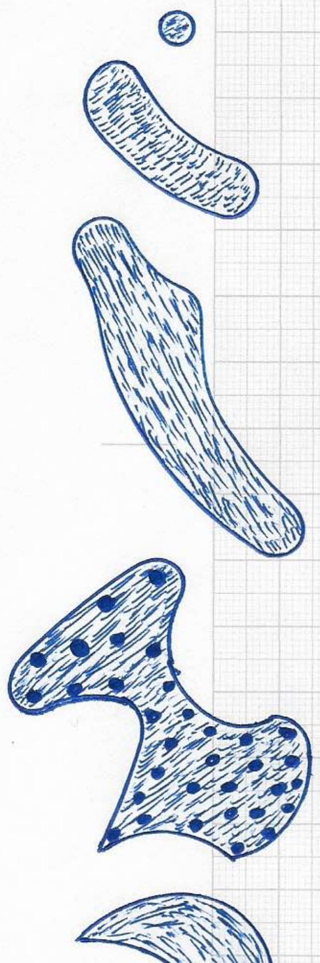
Cascade of region tests

This illustration shows that the progression of tests moves from simple tests to complex tests — where the simplest tests are "$\ast \notin {}^0R$?" and "$\ast \in {}^0r$?", and the most complex tests are "$\ast \notin {}^MR$?" and "$\ast \in {}^mr$?". This test progression therefore supports efficient classification, by first considering regions that are defined by relatively smaller numbers of half-spaces and less complicated Boolean expressions. Each test produces a TRUE (T) or FALSE (F) outcome. IF ALL TESTS YIELD THE OUTCOME 'F', THEN $\ast \in B$ (BOUNDARY LAYER). ...

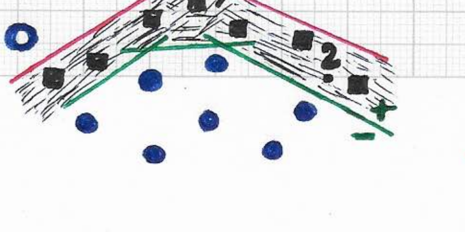
■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks:...

II) The "boundary layer" $\mathcal{B} = {}^{M}R - {}^{m}r$, where $M \neq m$, generally, is of particular significance for the classification process. Since one does not know, and cannot establish, the true, exact, analytically defined region R containing all allowed materials one may associate with the specific class of interest, one can merely use "plausible heuristics" for an approximation of R. For example, if one were to assume that R — a region in a high-dimensional space — is "topologically simple," i.e., a single compact region without holes, cavities, tunnels, ... (using a mathematically **imprecise**, colloquial description) that might have undergone a deformation, one could view the region R as a deformed high-dimensional (hyper-) ball. The sketches shown here provide examples of possible regions R in a two-dimensional space (**left**). One sketch shows samples '•' that lie inside R. WE CAN ONLY USE A SAMPLE SET {•} TO APPROXIMATE R AND, IN PARTICULAR, THE "BOUNDARY LAYER" $\mathcal{B}$. !



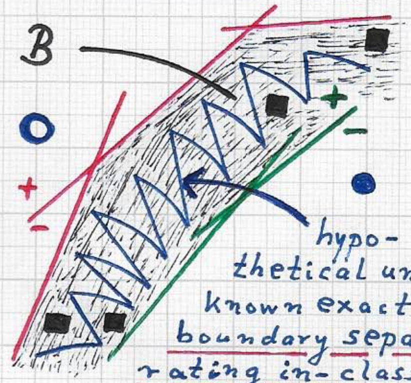
Viable unknown regions R to be approximated



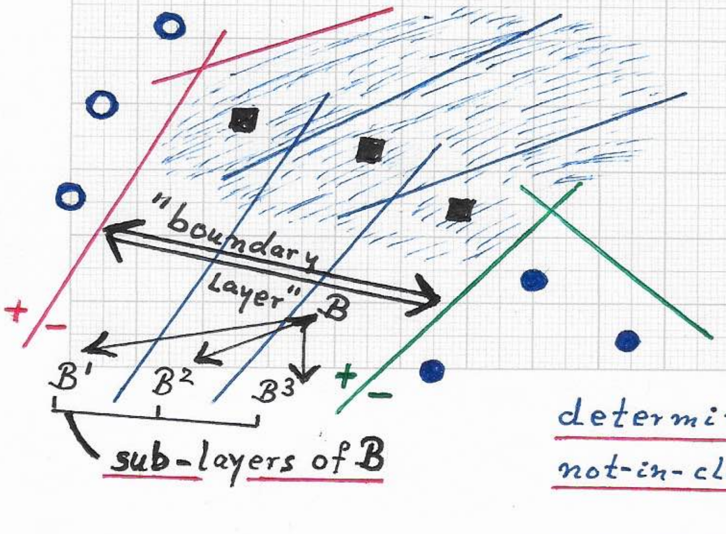
The left figure focuses on a part of $\mathcal{B}$. The set of points in $\mathcal{B}$, {■}, consists of points belonging and not belonging to the class. ...

■ OBJECT AND MATERIALS EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks:...



One can think of a multitude of "theoretical and practical arguments" for treating points ■ in the "boundary layer" B statistically: (i) The exact boundary separating in-class points ● and not-in-class points ○ can be extremely complicated, exhibiting substantial variation/oscillation and high-frequency behavior, making a super-high-fidelity representation impossible; (ii) this boundary could in fact be fractal in nature, thus not permitting a (finite) representation via analytical means; (iii) in a practical imaging, analysis and classification context, the degree of error and error propagation – and uncertainty – makes it necessary to treat the "boundary layer" B statistically, as the "coordinates" of a point ■ are imprecise.



• Note. For a statistical characterization of B one could, for example, consider 1000 "boundary layer" points ■ and – knowing their class membership status a priori – determine the fractions of in-class and not-in-class points, possibly in sub-layers of B.

...