Stratovan

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks:... The simplex algorithm (George Dantzig) is a classical algorithm used in linear programming that is related to solving our problem of establishing a definition of a region in high-dimensional space — a region $\mathcal{J}R$ or $\mathcal{J}r$ — using Boolean operations applied to half-spaces. We provide a high-level summary of necessary definitions, concepts and algebra.

12) First, we discuss the general alebraic representations of half-spaces via implicit linear hyper-plane definitions that split an N-dimensional space $(X_1, X_2, \ldots, X_N)^T$ into two half-spaces. The specific implicit linear hyper-plane equations for low-dimensional cases are the following:

- **N=1** $\quad c_0 + c_1 X_1 = 0 \quad$ Point
- **N=2** $\quad c_0 + c_1 X_1 + c_2 X_2 = 0 \quad$ Line
- **N=3** $\quad c_0 + c_1 X_1 + c_2 X_2 + c_3 X_3 = 0 \quad$ Plane
- **N=4** $\quad c_0 + c_1 X_1 + c_2 X_2 + c_3 X_3 + c_4 X_4 = 0 \quad$ Volume

- **N** $\quad c_0 + \sum_{k=1}^{N} c_k X_k = 0, \quad$ Hyper-Plane

$$L(\mathbf{X}) = L(X_1, \ldots, X_N) = c_0 + \sum_{k=1}^{N} c_k X_k .$$

The hyper-plane's outward (positive, +) normal "pointing into the positive half-space" is $\mathbf{n} = (c_1, \ldots, c_N)^T$, where $\|\mathbf{n}\| = \langle \mathbf{n}, \mathbf{n} \rangle^{1/2} = \left(\sum_{k=1}^{N} c_k^2\right)^{1/2} = 1$.

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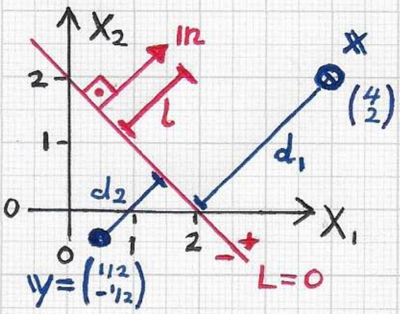
• Laplacian eigenfunctions and neural networks:...



The signed distance d between a point $\mathbb{x}=(x_1,\dots,x_N)^T$ and a hyperplane defined as $L(\mathbb{X})=0$ is

$$d = L(\mathbb{x})/l = \left(c_0+\sum_{k=1}^{N} c_k x_k\right)/l.$$

We consider the example shown in the figure:

$$L(X_1,X_2) = 1\cdot X_1 + 1\cdot X_2 - 2 \Rightarrow \mathbb{n}=(1,1)^T,\ l=\sqrt{2}$$

$$\Rightarrow d_1 = d(\mathbb{x}) = L(\mathbb{x})/l = (-2+4+2)/\sqrt{2} = 2\sqrt{2};$$

$$d_2 = d(\mathbb{y}) = L(\mathbb{y})/l = (-2+1/2-1/2)/\sqrt{2} = -\sqrt{2}.$$

The case $N=1$ can be used to simply illustrate the qualitative nature of the linear polynomial $L(\mathbb{X})=L(X_1)=c_0+c_1X_1$. For our purpose, i.e., using L merely as a "half-space defining representation," we are only concerned about the location in $\mathbb{X}$-space where $L(\mathbb{X})=0$, $L(\mathbb{X})<0$ and $L(\mathbb{X})>0$. We are not interested in the specific function value of $L(\mathbb{X})$. Therefore, it is possible, in our classification context, to apply an additional constraint, usually a linear condition, to the coefficients $c_0, c_1, \dots, c_N$. For $N=1$, we consider the following possibilities for constraining the value of c_0 to $c_0=1$ and $c_0=-1$. Thus,

$$L(X_1) = 1 + c_1X_1 \quad \text{and} \quad L(X_1) = -1 + c_1X_1.$$

...

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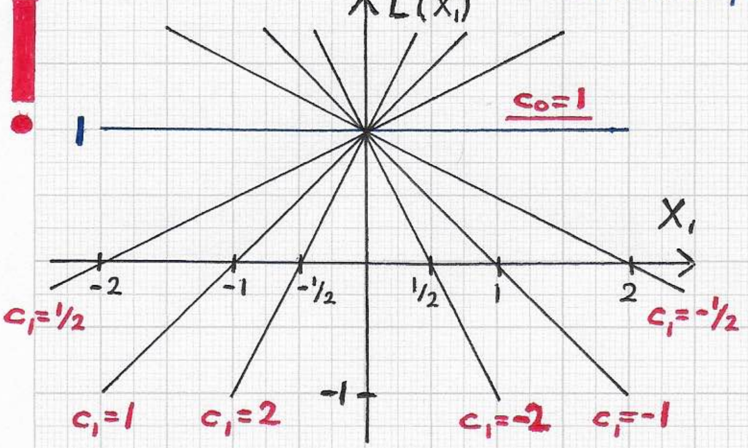
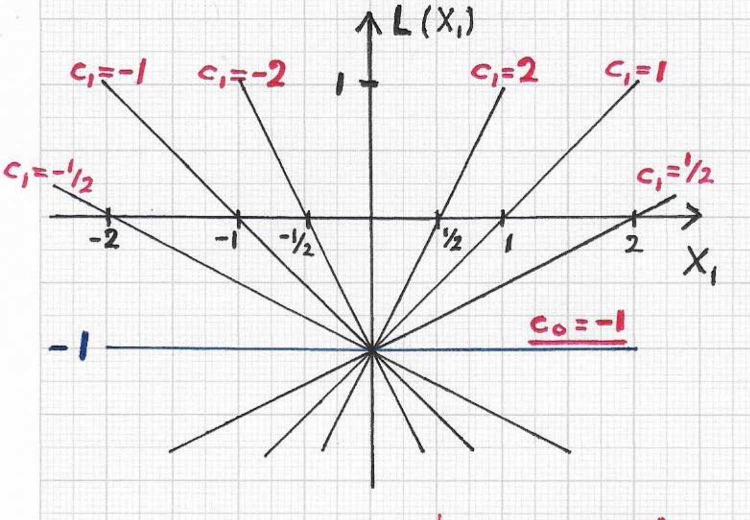
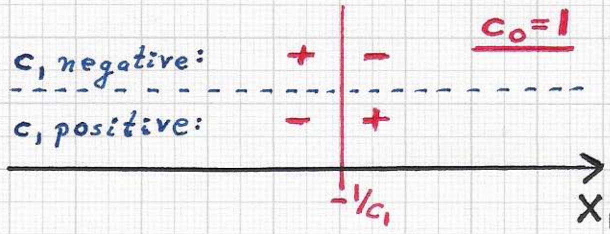
The two figures shown here (left) sketch families of graphs one obtains for different values of c_1 for the linear functions $L(X_1)=1+c_1X_1$ (top) and $L(X_1)=-1+c_1X_1$ (bottom). We must determine how these functions split the X_1-line into a negative and positive half-space, i.e., an interval where $L(X_1)\le 0$ and where $L(X_1)>0$. We consider the two cases:

$L>0$ / $L\le 0$



$L>0$ / $L\le 0$





The X_1-line is split into a positive and negative half-space at $X_1=-1/c_1$. This figure shows where $L>0$ (+) and $L\le 0$ (−). All possibilities are illustrated. $c_0=1$.

(i) $L(X_1)=1+c_1X_1=0$

$$\Rightarrow X_1=-1/c_1\ ,\ c_1\neq 0$$

$$\Rightarrow L(X_1)\begin{cases} >0\ , & -\infty<X_1<-1/c_1 \\ \le 0\ , & -1/c_1\le X_1<\infty \end{cases}$$

for $-\infty<c_1<0$;

$$L(X_1)\begin{cases} \le 0\ , & -\infty<X_1\le -1/c_1 \\ >0\ , & -1/c_1<X_1<\infty \end{cases}$$

for $0<c_1<\infty$.

(ii) $L(X_1)=-1+c_1X_1=0$

$$\Rightarrow X_1=1/c_1\ ,\ c_1\neq 0$$

$\Rightarrow$ ...

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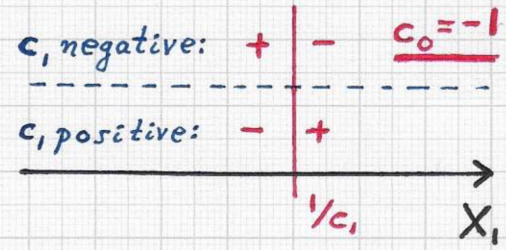


$\Rightarrow L(X_1) \begin{cases} > 0, & -\infty < X_1 < 1/c_1 \\ \le 0, & 1/c_1 \le X_1 < \infty \end{cases}$

for $-\infty < c_1 < 0$;

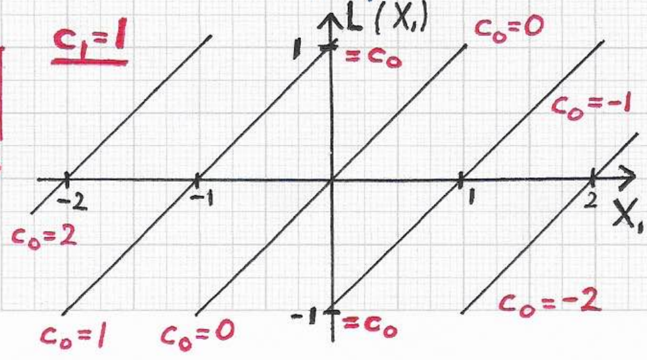
$L(X_1) \begin{cases} \le 0, & -\infty < X_1 \le 1/c_1 \\ > 0, & 1/c_1 < X_1 < \infty \end{cases}$

for $0 < c_1 < \infty$.

The X_1-line is split into a positive and negative half-space at $X_1 = 1/c_1$. This figure shows all possibilities for $c_0 = -1$.

The families of linear functions $L(X_1) = +/-1 + c_1 X_1$ are considered and analyzed in great detail here since they demonstrate the relationships between negative and positive half-spaces in the X_1-domain and the coefficients of a linear polynomial $L(X_1) = c_0 + c_1 X_1$. Another option for restricting the degree of freedom of this polynomial is based on prescribing the coefficient c_1 — defining the derivative/gradient of $L(X_1)$ — and use the value of the coefficient c_0 to determine the locations of negative and positive half-spaces in the X_1-domain. We consider the two cases $c_1 = 1$ and $c_1 = -1$, i.e., the linear polynomials $L(X_1) = c_0 + 1 \cdot X_1$ and $L(X_1) = c_0 - 1 \cdot X_1$. We discuss both cases:

$c_0 > 0$
$c_0 < 0$



(i) $L(X_1) = c_0 + 1 \cdot X_1 = 0$

$\Rightarrow X_1 = -c_0$, see left figure

$\Rightarrow \ldots$

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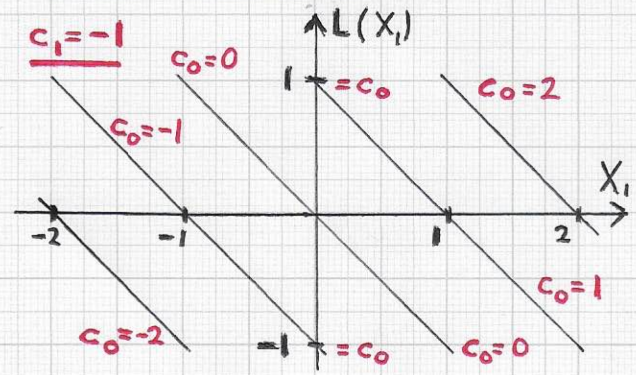
■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks: ...



$\frac{c_0 > 0}{c_0 < 0}$

$$\Rightarrow L(X_1) \begin{cases} \leq 0, & -\infty < X_1 \leq -c_0 \\ > 0, & -c_0 < X_1 < \infty \end{cases}$$

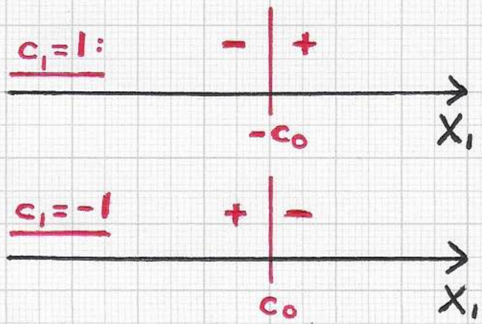
for $-\infty < c_0 < \infty$;

(ii) $L(X_1) = c_0 - 1 \cdot X_1 = 0$

$\Rightarrow X_1 = c_0$, see left figure

$$\Rightarrow L(X_1) \begin{cases} \leq 0, & c_0 \leq X_1 < \infty \\ > 0, & -\infty < X_1 < c_0 \end{cases}$$

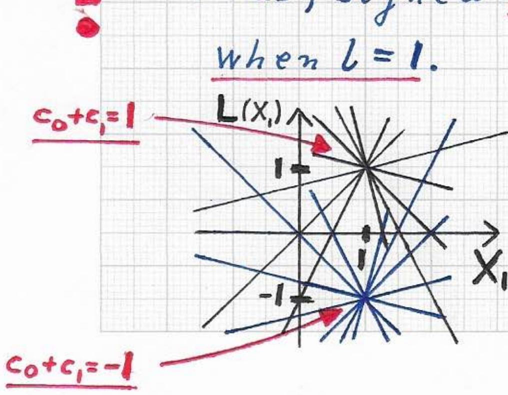
for $-\infty < c_0 < \infty$.



X_1-line split into half-spaces.

Based on the value of c_1, the X_1-line is split into a negative and positive half-space at $X_1 = -c_0$ (for $c_1 = 1$) and at $X_1 = c_0$ (for $c_1 = -1$), as illustrated in the two sketches (left).

• **Note**. The signed distance d between point $\mathbb{x}$ and hyper-plane $L(\mathbb{X}) = 0$ is $L(\mathbb{x})/l$, see p. 17 (3/20/2023). Here, l is the (absolute) length of the gradient vector of L. By setting $c_1 = +/-1$ (in the case $N = 1$), $l = 1$. Thus, signed distance computations are "simpler" when $l = 1$.



Yet another possibility to restrict the values of c_0 and c_1 is the enforcement of the linearity constraint $c_0 + c_1 = 1$ or $c_0 + c_1 = -1$. The families of associated functions are $L(X_1) = 1 + c_1(X_1 - 1) = 0$ and $L(X_1) = -1 + c_1(X_1 - 1) = 0$, respectively, see left figure.

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