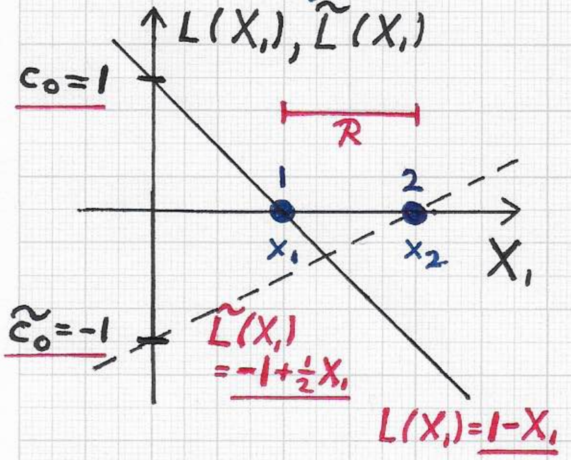
Stratovan

OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks:... Since the definition of a class region R is such a fundamental problem, we consider additional scenarios where the necessary negative half-spaces are established via linear polynomials. The scenario we discuss next de-



termines the wanted linear polynomials L and $\tilde{L}$ by using the following two families: $L(X_1) = 1 + c_1 X_1$ and $\tilde{L}(X_1) = -1 + \tilde{c}_1 X_1$. Again, instead of solving two univariate linear programming problems for calculating the optimal values for c_1 and $\tilde{c}_1$, we solve one problem for calculating the optimal tuple $(c_1, \tilde{c}_1)$, see figure (above). The four inequality conditions are

$$L(1) = 1 + c_1 \leq 0 \Rightarrow c_1 \leq -1 \;\wedge\; L(2) = 1 + 2c_1 \leq 0 \Rightarrow c_1 \leq -\tfrac{1}{2}$$
$$\wedge\; \tilde{L}(1) = -1 + \tilde{c}_1 \leq 0 \Rightarrow \tilde{c}_1 \leq 1 \;\wedge\; \tilde{L}(2) = -1 + 2\tilde{c}_1 \leq 0 \Rightarrow \tilde{c}_1 \leq \tfrac{1}{2}.$$

The lengths of the gradients of the polynomials are

$$l = (c_1{}^2)^{1/2} \Rightarrow l^2 = c_1{}^2 \text{ and } \tilde{l} = (\tilde{c}_1{}^2)^{1/2} \Rightarrow \tilde{l}^2 = \tilde{c}_1{}^2.$$

The following squared distances are defining the cost function:

$$d_1{}^2 = (L(1)/l)^2,\; d_2^2 = (L(2)/l)^2,\; \tilde{d}_1{}^2 = (\tilde{L}(1)/\tilde{l})^2,\; \tilde{d}_2{}^2 = (\tilde{L}(2)/\tilde{l})^2.$$

The resulting cost function to be minimized is the sum

$$S = d_1{}^2 + d_2{}^2 + \tilde{d}_1{}^2 + \tilde{d}_2{}^2 = ((1+c_1)^2 + (1+2c_1)^2)/l^2 + ((-1+\tilde{c}_1)^2 + (-1+2\tilde{c}_1)^2)/\tilde{l}^2$$
$$= (2 + 6c_1 + 5c_1{}^2)/c_1{}^2 + (2 - 6\tilde{c}_1 + 5\tilde{c}_1{}^2)/\tilde{c}_1{}^2 \rightarrow min$$
$$\Rightarrow (c_1, \tilde{c}_1) = (-1, \tfrac{1}{2}).$$

The figure (above) shows the computed functions L and $\tilde{L}$.

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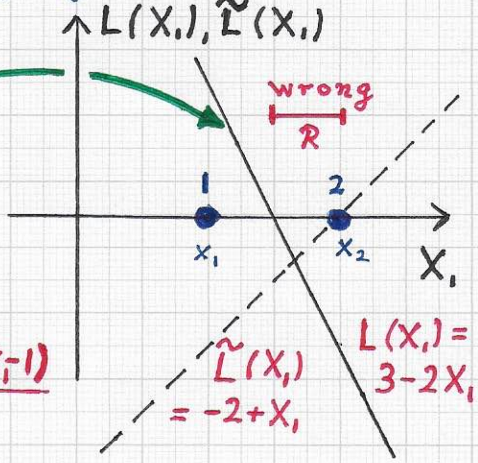
OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks:... • **Note.** It is important to keep in mind that "meaningful, viable and acceptable solutions" are not always possible when solving a linear programming problem — e.g., as a consequence of false inequalities, contradictary inequality conditions or inequalities that cannot be satisfied simultaneously. We consider an example where we use the same X_1-input data as used in the last example, but where we constrain the linear polynomials L and $\tilde{L}$ in yet another way: $c_0 + c_1 = 1$ and $\tilde{c}_0 + \tilde{c}_1 = -1$. Thus, we use the polynomial families $L(X_1) = (1-c_1) + c_1 X_1 = 1 + c_1 (X_1 - 1)$ and $\tilde{L}(X_1) = (-1-\tilde{c}_1) + \tilde{c}_1 X_1 = -1 + \tilde{c}_1 (X_1 - 1)$.

$L(X_1)$ "should be" $L(X_1) = 1 - 10^{10}(X_1 - 1)$



The four inequality conditions are $\boxed{L(1) \le 0 \Rightarrow 1 \le 0}$ ⊕ $\wedge\ L(2) \le 0 \Rightarrow c_1 \le -1$ $\wedge\ \tilde{L}(1) \le 0 \Rightarrow -1 \le 0$ $\wedge\ \tilde{L}(2) \le 0 \Rightarrow \tilde{c}_1 \le 1$.

Using the polynomials L and $\tilde{L}$ and evaluating them for $X_1 = 1$ and $X_1 = 2$, one obtains the cost function sum:

$$S = d_1^2 + d_2^2 + \tilde{d}_1^2 + \tilde{d}_2^2 = \left(1^2 + (1+c_1)^2\right)/c_1^2 + \left((-1)^2 + (-1+\tilde{c}_1)^2\right)/\tilde{c}_1^2$$
$$= (2 + 2c_1 + c_1^2)/c_1^2 + (2 - 2\tilde{c}_1 + \tilde{c}_1^2)/\tilde{c}_1^2 \rightarrow \min$$
$$\Rightarrow (c_1, \tilde{c}_1) = (-2, 1). \Rightarrow L(X_1) = 1 - 2(X_1 - 1),\ \tilde{L}(X_1) = -1 + (X_1 - 1).$$

THESE POLYNOMIALS ARE "WRONG": L and $\tilde{L}$ are both negative for the region/interval $[\frac{3}{2}, 2] = R$ — which should be $R = [1, 2]$. ⊕ THE FIRST INEQUALITY IS FALSE!

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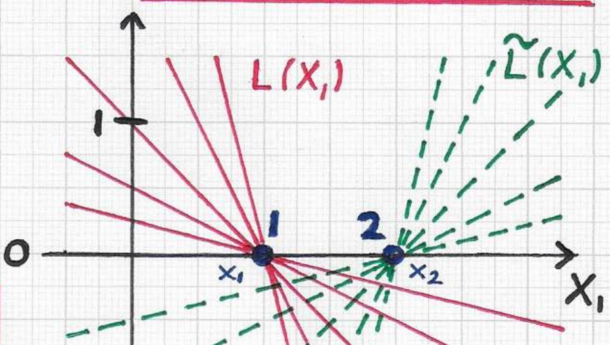
OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks:... • **Note.** The covered examples concerning linear programming problems are "not fully linear." While the half-space-defining polynomials L and $\tilde{L}$ are indeed linear polynomials, we used cost functions involving "sums of squared distances" and "squares of gradient lengths," for example. Thus, we are really discussing a problem that is "slightly more general" than a purely linear problem.

It is relevant and of interest to explore the more general setting for the simple example considered: We do not constrain the coefficients $c_0, c_1, \tilde{c}_0$ and $\tilde{c}_1$, and determine entire allowable families of allowable solutions for L and $\tilde{L}$. The figure shown here (left) illustrates the allowable, general linear polynomials; they are:



$$L(X_1) = c_1(X_1 - 1) = -c_1 + c_1 X_1 ,$$
$$-\infty < c_1 < 0 ;$$
$$\tilde{L}(X_1) = \tilde{c}_1(X_1 - 2) = -2\tilde{c}_1 + \tilde{c}_1 X_1 ,$$
$$0 < \tilde{c}_1 < \infty .$$

In this simple example, it becomes evident that c_1 ($\tilde{c}_1$) must be and can be any negative (positive) real number. Further, L and $\tilde{L}$ only have one degree of freedom, since $c_0 = -c_1$ and $\tilde{c}_0 = -2\tilde{c}_1$. We can constrain the values of c_1 and $\tilde{c}_1$. ...

OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks:... It is now possible to formally establish and solve the general problem:

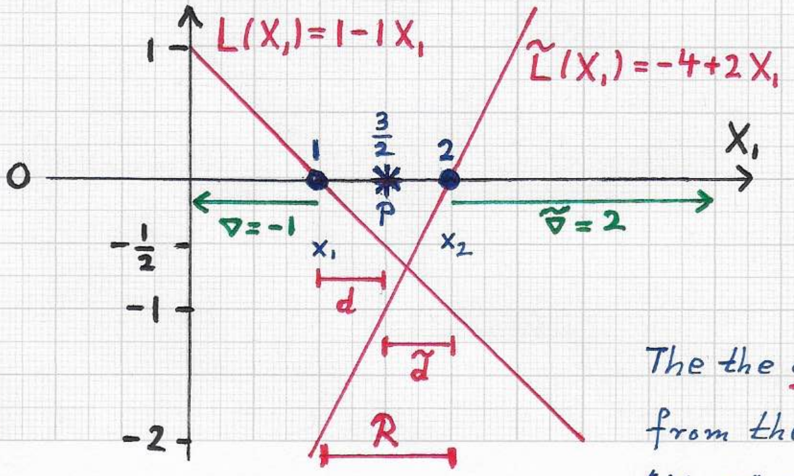
$$L(1)=0 \Rightarrow 0\le 0 \wedge L(2)=c_1 \Rightarrow c_1 \le 0 \wedge \tilde{L}(1)=-\tilde{c}_1 \Rightarrow \tilde{c}_1 \ge 0 \wedge \tilde{L}(2)=0 \Rightarrow 0 \le 0.$$

The squared l-values of L and $\tilde{L}$ are $l^2=c_1^2$ and $\tilde{l}^2=\tilde{c}_1^2$.
We now evaluate the two linear polynomial for all (=two) x_i-values ($x_1=1$, $x_2=2$) to define the cost function sum:

$$\underline{S} = d_1^2+d_2^2+\tilde{d}_1^2+\tilde{d}_2^2 = (0+c_1^2)/c_1^2+((-\tilde{c}_1)^2+0)/\tilde{c}_1^2$$
$$= c_1^2/c_1^2 + \tilde{c}_1^2/\tilde{c}_1^2 = 2 \rightarrow min$$

$\Rightarrow$ ONE CAN CHOOSE THE VALUES OF c_1 AND $\tilde{c}_1$ FOR THE TUPLE $(c_1, \tilde{c}_1)$ ARBITRARILY, as long as $-\infty < c_1 < 0 \wedge 0 < \tilde{c}_1 < \infty$.

Thus, the two linear polynomials and their coefficients were, in this example, constrained from the very beginning that made it impossible/made it irrelevant to further constrain the values of c_1 and $\tilde{c}_1$. The cost function $S(c_1, \tilde{c}_1)=2$, i.e., is constant, and is minimal for all values of c_1 and $\tilde{c}_1$.

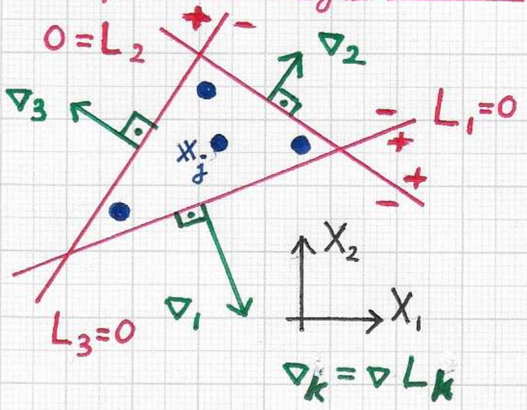


The left figure shows a randomly chosen pair of allowable polynomials L and $\tilde{L}$. Their gradients are $\nabla=\nabla L=-1$ and $\tilde{\nabla}=\nabla\tilde{L}=2$.
The the distance values of $p=\frac{3}{2}$ from the left (right) boundary $X_1=1$ ($X_1=2$) is $d=L(p)/l=-\frac{1}{2}/1=(-\frac{1}{2})$ ($\tilde{d}=\tilde{L}(p)/\tilde{l}=-1/2=(-\frac{1}{2})$). ...

OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks: ...

• **Note.** The last simple "general" example makes clear that the gradients/gradient vectors of L and $\tilde{L}$ are "pointing away from the region R": $\nabla L = c_1$, $c_1 \in (-\infty, 0)$, is the "vector" (c_1) that points from $X_1 = 1$ to $X_1 = -\infty$; $\nabla \tilde{L} = \tilde{c}_1$, $\tilde{c}_1 \in (0, \infty)$, is the "vector" $(\tilde{c}_1)$ that points from $X_1 = 2$ to $X_1 = \infty$. These two "vectors" are called ∇ and $\tilde{\nabla}$ in the figure shown on the previous page, i.e., $\nabla = (-1)$ and $\tilde{\nabla} = (2)$ for the chosen two polynomials. The relevance of this fact for the general multivariate setting is the following: In the general setting, a region R is defined by the negative half-spaces of linear polynomials $L_1(\mathbb{X}), L_2(\mathbb{X}), L_3(\mathbb{X}), \ldots$; all gradient vectors $\nabla L_1, \nabla L_2, \nabla L_3, \ldots$ must "point away from R." These vectors are constant, for each linear polynomial, and are perpendicular to all contours/isolines, for each linear polynomial. The left figure illustrates this characteristic for three polynomials $L_k(X_1, X_2)$.



It has also been seen that the length l of the gradient vector is irrelevant — usually — for our purposes. Thus, we can constrain gradients of polynomials $L(\mathbb{X}) = c_0 + \sum_{i=1}^{N} c_i X_i$ and set $l^2 = \sum_{i=1}^{N} c_i^2 = 1$, thereby simplifying some computations.

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