OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

- Laplacian eigenfunctions and neural networks:... We can generalize the computation of the values c_0^i, $i=1,2$, to the bivariate example (see last two pages), to calculate c_0^i, $i=1\ldots4$. The normalized, unit gradient vectors, i.e., $\mathbf{e}_1$, $-\mathbf{e}_1$, $\mathbf{e}_2$ and $-\mathbf{e}_2$, define functions – linear polynomials – having these unit vectors as their gradients and having function value 0 at $(X_1, X_2) = (0,0)$:

$$F_1 = \sqrt{2}/2\,X_1 + \sqrt{2}/2\,X_2,\quad F_2 = -\sqrt{2}/2\,X_1 + \sqrt{2}/2\,X_2,\quad F_3 = -\sqrt{2}/2\,X_1 - \sqrt{2}/2\,X_2,$$
$$F_4 = \sqrt{2}/2\,X_1 - \sqrt{2}/2\,X_2 .$$

The corresponding linear polynomials to be determined are $L_i(X_1, X_2) = c_0^i + F_i(X_1, X_2)$, $i=1\ldots4$. Following the univariate example, we must evaluate $F_i(X_1, X_2)$, $i=1\ldots4$, for all given points $\mathbf{x}_j$, $j=1\ldots4$, to determine, for each function F_i, its minimal and maximal values, called F_i^{min} and F_i^{max}. The resulting minimal and maximal values are provided in the left table. Thus, the L_i functions are:

	$\mathbf{x}_1$	$\mathbf{x}_2$	$\mathbf{x}_3$	$\mathbf{x}_4$	F_i^{min}	F_i^{max}
F_1	$\frac{7}{2}\sqrt{2}$	$\frac{7}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{2}$	$\frac{7}{2}\sqrt{2}$
F_2	$-\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{2}$	$-\frac{1}{2}\sqrt{2}$	$-\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{2}$
F_3	$-\frac{7}{2}\sqrt{2}$	$-\frac{7}{2}\sqrt{2}$	$-\frac{1}{2}\sqrt{2}$	$-\frac{1}{2}\sqrt{2}$	$-\frac{7}{2}\sqrt{2}$	$-\frac{1}{2}\sqrt{2}$
F_4	$\frac{1}{2}\sqrt{2}$	$-\frac{1}{2}\sqrt{2}$	$-\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{2}$	$-\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{2}$

$$L_1 = -F_1^{max} + F_1 = -\tfrac{7}{2}\sqrt{2} + F_1$$
$$L_2 = -F_2^{max} + F_2 = -\tfrac{1}{2}\sqrt{2} + F_2$$
$$L_3 = -F_3^{max} + F_3 = +\tfrac{1}{2}\sqrt{2} + F_3$$
$$L_4 = -F_4^{max} + F_4 = -\tfrac{1}{2}\sqrt{2} + F_4$$

The values of the constant terms of the linear polynomials are $c_0^i = -F_i^{max}$, $i=1\ldots4$. In this bivariate example, the values are $c_0^1 = -\frac{7}{2}\sqrt{2}$, $c_0^2 = -\frac{1}{2}\sqrt{2}$, $c_0^3 = +\frac{1}{2}\sqrt{2}$, $c_0^4 = -\frac{1}{2}\sqrt{2}$.

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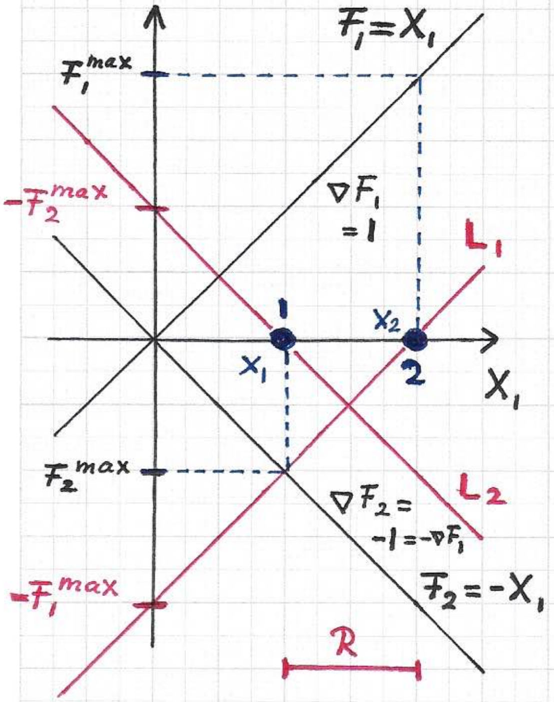
Bernd Hamann
Stratovan

4/6/2023

OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

- Laplacian eigenfunctions and neural networks: ...



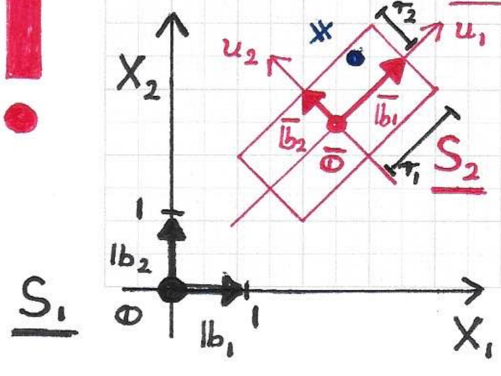
	X_1	x_2	F_i^{min}	F_i^{max}
F_1	1	2	1	2
F_2	-1	-2	-2	-1

We can employ the same formal approach for the univariate – and multivariate – case to determine the values of the constants of the linear polynomials. The left figure and table show the data for the simple univariate example involving only x_1 and x_2:

$$F_1(X_1) = X_1 \ , \ F_2(X_1) = -X_1 \ ;$$
$$F_1^{max} = F_1(2) = 2 \ ,$$
$$F_2^{max} = F_2(1) = -1 \ ;$$
$$L_1 = -F_1^{max} + F_1 = -2 + X_1 \ ,$$
$$L_2 = -F_2^{max} + F_2 = +1 + X_1 \ .$$

- **Note.** An important fact concerning this method for determining linear polynomials is the LINEAR increase of the number of needed polynomials $L_i(\mathbb{X})$ with increasing domain dimension.

- **USING THE LINEAR TRANSFORMATION – CONCATENATION OF SCALING, ROTATION AND TRANSLATION – TO DESIGN AN "INSIDE-ELLIPSE(-ELLIPSOID)-BOUNDED-REGION TEST"** First, we must review how the coordinates of a point $\mathbb{X}$ are related when expressing $\mathbb{X}$ to two coordinate systems (left figure).



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■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks: … Specifically, we must define the linear transformation that maps a coordinate system S_1 — a right-handed system with origin $\mathbb{O}$ and orthogonal unit, normalized basis vectors $\mathbb{b}_1$ and $\mathbb{b}_2$ — to a coordinate system S_2 — with origin $\overline{\mathbb{O}}$ and orthogonal basis vectors $\overline{\mathbb{b}}_1$ and $\overline{\mathbb{b}}_2$. We can understand this transformation as a concatenation of **(i)** scaling basis vectors $\mathbb{b}_1$ and $\mathbb{b}_2$ by scaling factors r_1 and r_2, respectively, to scale the unit length of these vectors to the desired lengths r_1 and r_2 of $\overline{\mathbb{b}}_1$ and $\overline{\mathbb{b}}_2$; **(ii)** applying a rotation matrix to the vectors resulting from step (i), to establish the desired orientations of $\overline{\mathbb{b}}_1$ and $\overline{\mathbb{b}}_2$; and **(iii)** translating the original origin $\mathbb{O}$ to the new origin $\overline{\mathbb{O}}$. We adopt the concept of homogeneous coordinates and write the point $\mathbb{x}$ as column vector $(X_1, X_2, 1)^T$. The individual matrices of the three transformations can also be written in homogeneous form; they are:

$$\underline{S} = \begin{bmatrix} r_1 & 0 & 0 \\ 0 & r_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \underline{R} = \begin{bmatrix} [\mathbb{e}_1] & [\mathbb{e}_2] & \begin{matrix} 0 \\ 0 \end{matrix} \\ 0 & 0 & 1 \end{bmatrix}, \quad \underline{T} = \begin{bmatrix} \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} & [\mathbb{t}] \\ \begin{matrix} 0 & 0 \end{matrix} & 1 \end{bmatrix},$$

where $\mathbb{e}_1$ and $\mathbb{e}_2$ are the normalized, unit "versions" of $\overline{\mathbb{b}}_1$ and $\overline{\mathbb{b}}_2$, and $\mathbb{t}$ is the translation vector $\mathbb{t} = \overline{\mathbb{O}} - \mathbb{O}$. Thus, the transformation matrix M mapping system S_1 to S_2 is

$$\underline{M = T \cdot R \cdot S}.$$

…

OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks:... We are interested in the representation of a point $\mathbb{x}$ relative to the coordinate system S_2 that is the result of mapping the origin and the orthonormal basis vectors to the origin and the orthogonal (not normalized) basis vectors of the second system S_2 — "mapping S_1 to S_2." A point $\mathbb{x}$ has the coordinate representation $_1\mathbb{x}$ relative to system S_1, and we must calculate $_2\mathbb{x}$, i.e., the coordinate representation of $\mathbb{x}$ relative to system S_2. Since we know that $M = T \cdot R \cdot S$ maps S_1 to S_2, we can use a fact concerning this linear coordinate transformation: The representation $_2\mathbb{x}$ is obtained by applying T^{-1}, then R^{-1}, then S^{-1} to $_1\mathbb{x}$, i.e., applying the inverse transformations of T, R and S in reverse concatenation order to $_1\mathbb{x}$. The inverse translation operation uses the vector $-\mathbb{t}$; the inverse rotation operation uses the matrix that is the transpose of R; and the inverse scaling operation uses the inverse factors $1/r_1$ and $1/r_2$. The homogeneous matrices are:

$$T^{-1} = \begin{bmatrix} 1 & 0 & \multirow{2}{*}{$[-\mathbb{t}]$} \\ 0 & 1 & \\ 0 & 0 & 1 \end{bmatrix}, \quad R^{-1} = \begin{bmatrix} [\ \mathbb{e}_1^T\ ] & 0 \\ [\ \mathbb{e}_2^T\ ] & 0 \\ 0 \quad 0 & 1 \end{bmatrix}, \quad S^{-1} = \begin{bmatrix} 1/r_1 & 0 & 0 \\ 0 & 1/r_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ where}$$

$\mathbb{e}_1^T = (e_1^1, e_2^1)$ and $\mathbb{e}_2^T = (e_1^2, e_2^2)$ are the normalized (eigen)vectors $\mathbb{e}_1$ and $\mathbb{e}_2$ — written as rows. The needed inverse transformation matrix is $M^{-1} = S^{-1} R^{-1} T^{-1}$.

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OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

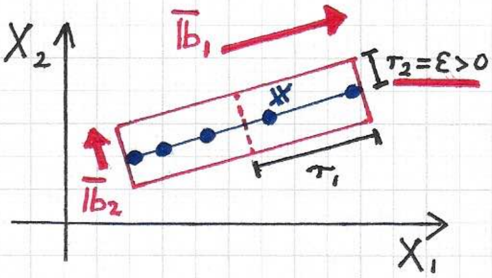
- Laplacian eigenfunctions and neural networks: ... Using the notation $\mathbf{t} = (t_1, t_2)^T$ for the translation vector, one obtains

$$\underline{\underline{M^{-1}}} = \begin{bmatrix} 1/r_1 & 0 & 0 \\ 0 & 1/r_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} e_1^1 & e_2^1 & 0 \\ e_1^2 & e_2^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & -t_1 \\ 0 & 1 & -t_2 \\ 0 & 0 & 1 \end{bmatrix} = \underline{\underline{\begin{bmatrix} e_1^1/r_1 & e_2^1/r_1 & -(t_1 e_1^1 + t_2 e_2^1)/r_1 \\ e_1^2/r_2 & e_2^2/r_2 & -(t_1 e_1^2 + t_2 e_2^2)/r_2 \\ 0 & 0 & 1 \end{bmatrix}}} .$$

Therefore, the representation $_2\ast$ is given as

$$\underline{\underline{{}_2\ast = M^{-1}\, {}_1\ast}} .$$

- **Note.** In degenerate point set configurations, it is possible that eigenvalues of the C (covariance) matrix are zero, leading to r_i-values that are zero. These "nearly zero" or zero values of "radii r_i" must be treated as special cases when calculating M^{-1} and $_2\ast$. In other words, a given point set $\{\bullet\}$ embedded in a space of dimension D can have a "point-set-inherent dimension" that is smaller than D. The left figure shows an example where $D=2$ and a point set $\{\bullet\}$ lies on a line. The covariance matrix C of this point set would have one eigenvalue with value zero. For practical purposes, one would assign a value ε to the associated "radius" (r_2) and also assign the corresponding basis vector of length ε to the orthogonal second basis vector set $\{\mathbb{b}_1, \mathbb{b}_2\}$.



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