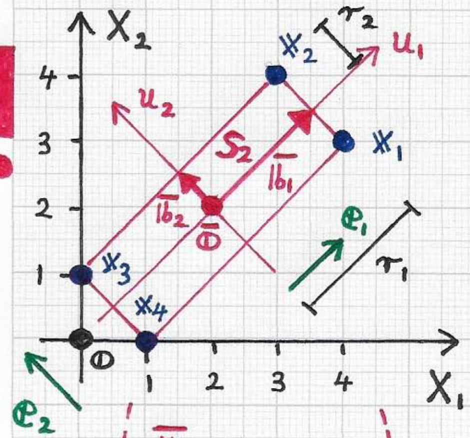
OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

- Laplacian eigenfunctions and neural networks: ...



$$\begin{pmatrix} \mathbf{b}_1 = r_1\, \mathbf{e}_1 \\ \mathbf{b}_2 = r_2\, \mathbf{e}_2 \end{pmatrix}$$

We provide simple numerical example for the described linear coordinate transformation by using the data shown in the left figure. Here,

$$\mathbf{t} = \bar{o} - o = (2,2)^T,$$
$$\mathbf{b}_1 = \left(\tfrac{3}{2}, \tfrac{3}{2}\right)^T,$$
$$\mathbf{b}_2 = \left(-\tfrac{1}{2}, \tfrac{1}{2}\right)^T,$$
$$r_1 = \tfrac{3}{2}\sqrt{2}, \quad r_2 = \tfrac{1}{2}\sqrt{2},$$
$$\mathbf{e}_1 = \left(\tfrac{1}{2}\sqrt{2}, \tfrac{1}{2}\sqrt{2}\right)^T,$$
$$\mathbf{e}_2 = \left(-\tfrac{1}{2}\sqrt{2}, \tfrac{1}{2}\sqrt{2}\right)^T.$$

The individual homogeneous transformation matrices are

$$S^{-1} = \begin{bmatrix} \frac{1}{3}\sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad R^{-1} = \begin{bmatrix} \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} & 0 \\ -\frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } T^{-1} = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}.$$

Thus, the resulting matrix $M^{-1} = S^{-1} R^{-1} T^{-1}$ is

$$M^{-1} = \begin{bmatrix} 1/3 & 1/3 & -4/3 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

We can apply this matrix to the four points x_i, $i=1\ldots4$, for example. We write the points as columns of a point matrix P, using homogeneous coordinates for the points. The given points x_i are represented relative to system S_1, i.e., $(X_1, X_2)^T$ coordinate columns define them. The transformation yields

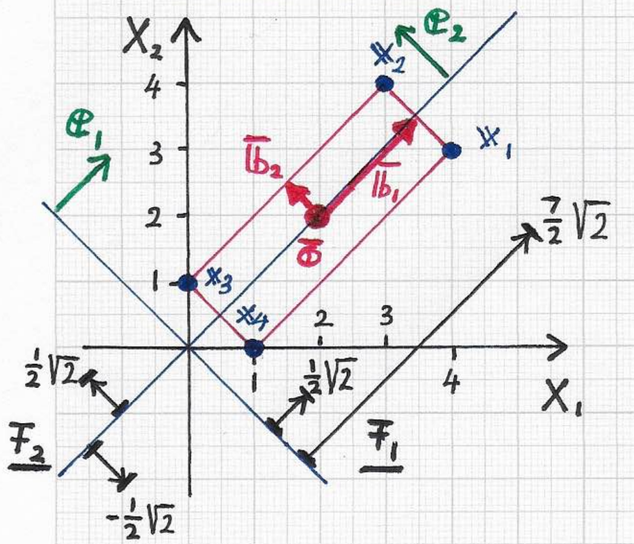
$$U = M^{-1} P = \begin{bmatrix} 1/3 & 1/3 & -4/3 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 3 & 0 & 1 \\ 3 & 4 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

The columns of matrix U represents the $(u_1, u_2)^T$ representations of the points relative to the second system S_2.

...

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks: ...



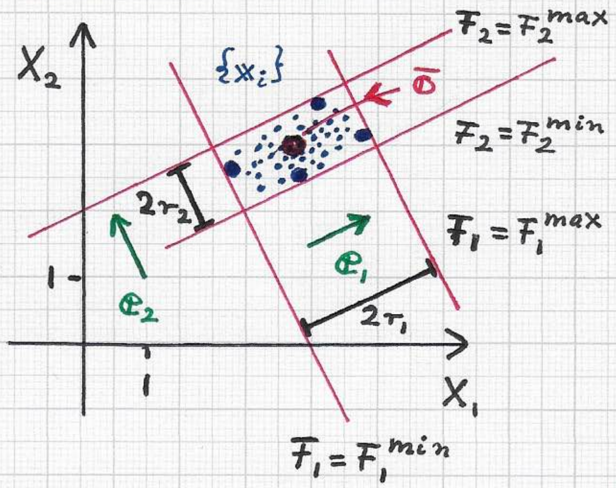
• **Note.** It is important to understand how the second coordinate system S_2 is constructed from a given set of points $\{\mathbb{x}_i\}$. The left figure once again shows the special case of a point set consisting of four points defining the vertices of a rectangle in the plane, i.e., the rectangle that defines the (minimal) bounding box of the point set. In general, the given point set contains any number of points with arbitrary positions in the plane — and one would have to determine the four vertices of the (minimal) bounding box RELATIVE TO THE LINEAR POLYNOMIALS $F_i(X_1, X_2)$ AS DESCRIBED ON PAGE 11 (4/5/2023). The figure refers to these underlying polynomials $F_1 = F_1(X_1, X_2)$ and $F_2 = F_2(X_1, X_2)$. The unit eigendirection vectors $\mathbb{e}_1$ and $\mathbb{e}_2$ are the two orthonormal vectors derived from the two orthogonal eigenvectors obtained from the **C** (covariance) **matrix**[⊗], considering all points in $\{\mathbb{x}_i\}$. They define the orientation of the (minimal) bounding box.

⊗ after "mean subtraction"

• • •

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• **Laplacian eigenfunctions and neural networks: ...** The normalized eigendirections $\mathbf{e}_1$ and $\mathbf{e}_2$ serve as gradients for the linear polynomials $F_1(X_1, X_2)$ and $F_2(X_1, X_2)$ that satisfy $F_1(0,0) = F_2(0,0) = 0$. A pair of isolines of each of these two polynomials defines the boundary line segments of the (minimal) bounding box (left figure).



We can now calculate needed values defining system S_2:

$$2r_1 = F_1^{max} - F_1^{min} = 7/2\sqrt{2} - 1/2\sqrt{2} = 3\sqrt{2},$$

$$2r_2 = F_2^{max} - F_2^{min} = 1/2\sqrt{2} - (-1/2)\sqrt{2} = \sqrt{2}.$$

$$\begin{aligned}\bar{\Phi} &= \tfrac{1}{2}(F_1^{min} + F_1^{max})\,\mathbf{e}_1 + \tfrac{1}{2}(F_2^{min} + F_2^{max})\,\mathbf{e}_2 \\ &= \tfrac{1}{2}(1/2\sqrt{2} + 7/2\sqrt{2})\,(\sqrt{2}/2, \sqrt{2}/2)^T \\ &\quad + \tfrac{1}{2}(-1/2\sqrt{2} + 1/2\sqrt{2})\,(-\sqrt{2}/2, \sqrt{2}/2)^T \\ &= 2\sqrt{2}\,(\sqrt{2}/2, \sqrt{2}/2)^T + 0\,(-\sqrt{2}/2, \sqrt{2}/2)^T \\ &= (2, 2)^T.\end{aligned}$$

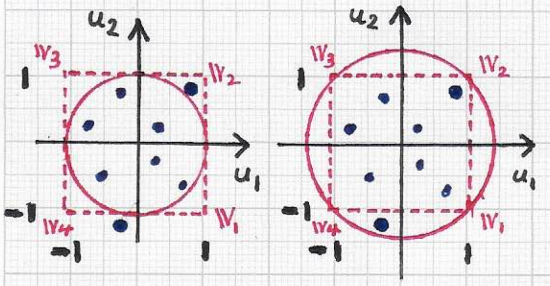
(See p. 11 (4/5/2023) for the used numerical values.)

• **Note.** It is important to keep in mind that $\bar{\Phi}$ IS NOT THE MEAN of the given points in the set $\{x_i\}$. The coordinates of $\bar{\Phi}$ are the coordinates of the vertices/corner points of the constructed (minimal) bounding box of the point set $\{x_i\}$. We do not use the mean $\bar{x}$ of the given point set, since it can be far away from $\bar{\Phi}$ for some distributions (left figure).

• • •

■ OBJECT AND MATERIAL EIGENFUNCTIONS - Cont'd.

• Laplacian eigenfunctions and neural networks: ...



Inscribed/circumscribed circles

We must recall what the (main) purpose of this linear coordinate transformation is:

i) We must determine whether a "point (most likely) does/does not belong to a class."

ii) We are provided with a large set of "sample points" belonging to a specific class, and we are given one "unclassified point" that must be classified.

iii) The set of "sample points" or the union of subsets of these "sample points" define class-specific regions in space that define the class.

iv) For a given "unclassified point," one must determine, with a high degree of certainty, whether it is inside one of the given class-specific regions or not.

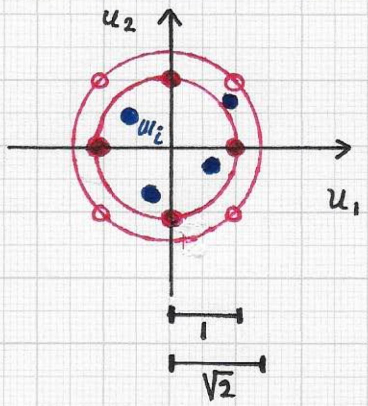
v) Therefore, the designed algorithm for performing the required "inside-outside test" for an "unclassified point" and class-specific regions must be as efficient as possible.

vi) The described transformation $M^{-1} = S^{-1} R^{-1} T^{-1}$ maps the (minimal) bounding box of $\{x_i\}$ to the square with vertices $v_1 = (1,-1)^T$, $v_2 = (1,1)^T$, $v_3 = (-1,1)^T$ and $v_4 = (-1,-1)^T$. The figure on this page shows this square and its associated inscribed and circumscribed circles, together with points '•' lying inside the square.

...

OBJECT AND MATERIAL EIGENFUNCTIONS - Cont'd.

• Laplacian eigenfunctions and neural networks:...



Inscribed and circumscribed circles together.

vii) The inscribed and circumscribed circles define two regions in the (u_1, u_2)-plane (left figure). The circles themselves are the boundaries of these two regions. The inscribed circle passes through '•', the $\mathbb{u}$ points $(1,0)^T$, $(0,1)^T$, $(-1,0)^T$ and $(0,-1)^T$. It is defined by $(u_1)^2 + (u_2)^2 = 1$. The circumscribed circle passes through '∘', the $\mathbb{u}$ points $(1,-1)^T$, $(1,1)^T$, $(-1,1)^T$ and $(-1,-1)^T$. It is defined by $(u_1)^2 + (u_2)^2 = 2$.

viii) When applying the linear transformation M^{-1} to the given point set $\{\mathbb{x}_i = (X_1^i, X_2^i)^T\}$, each point $\mathbb{x}_i$ is mapped to its image point $\mathbb{u}_i = (u_1^i, u_2^i)^T$. The above figure includes four points '•', image points $\mathbb{u}_i$; these four points all lie inside the square $[-1,1]^2$, BUT ONE POINT '•' LIES IN THE REGION BETWEEN THE TWO CIRCLES.

ix) In (X_1, X_2)-space, four linear polynomials are used to determine whether an "unclassified point" $\mathbb{x}$ lies inside the (minimal) bounding box of $\{\mathbb{x}_i\}$. Concerning (u_1, u_2)-space, all points $\{\mathbb{u}_i\}$ satisfy a single CIRCUMSCRIBED CIRCLE TEST: $(u_1^i)^2 + (u_2^i)^2 \leq 2$. IT IS GENERALLY NOT THE CASE THAT ALL $\mathbb{u}_i$ POINTS ALSO SATISFY $(u_1^i)^2 + (u_2^i)^2 \leq 1$.

•••