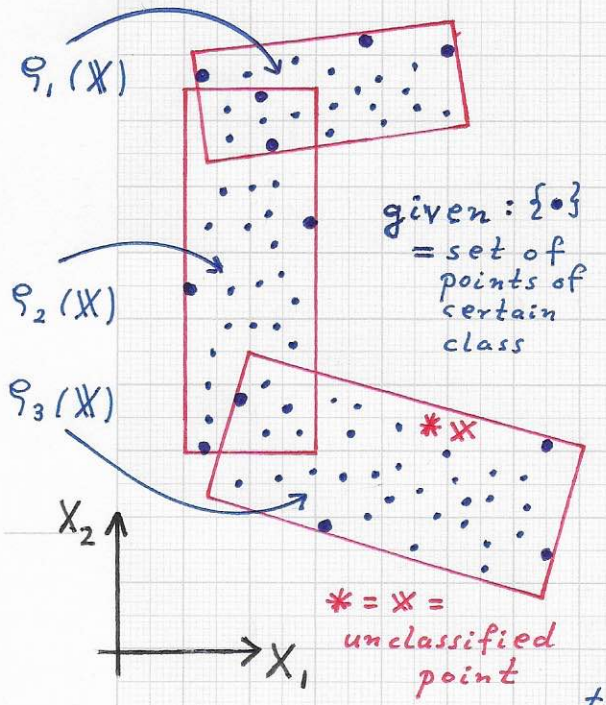
OBJECT AND MATERIAL EIGENFUNCTIONS - Cont'd.

• Laplacian eigenfunctions and neural networks!...



At this point, we return the discussion to the issues i.), ii.),..., x.) described on pp. 19-21 (4/12/2023 - 4/13/2023). On a high and abstract level, one can summarize the task to be addressed as follows: Given a set of points '•', all belonging to the same material class, embedded in D-dimensional space and one unclassified point ✳, estimate a number that makes it possible to determine the probability of ✳ belonging to the same material class. The figure on this page sketches a scenario where a set {•} is defining a point distribution in the plane where the points "lie in a ℂ-like region." This distribution is not "simple," and the methods described would apply repeated point set splitting / subdivision to generate a set of minimal bounding boxes of point subsets until the union of these bounding box regions is a "tight and good approximation" of the ℂ-like region, as shown in the figure. The figure refers to density functions $\varrho_i(\mathbb{X})$, $i = 1, 2, 3$, associated with the three bounding box regions. These local densities $\varrho_i(\mathbb{X})$ must be blended to define a global density function $\varrho(\mathbb{X})$.

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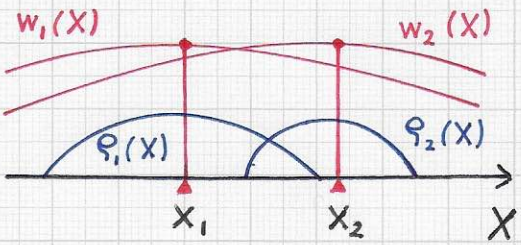
OBJECT AND MATERIAL EIGENFUNCTIONS - Cont'd.

• Laplacian eigenfunctions and neural networks...



Scattered data interpolation is usually concerned with the interpolation of a finite set of discrete function values f_i given for corresponding sites $\mathbb{X}_i$ in a D-dimensional domain space. A generalization of the scattered data interpolation problem considers the interpolation of functions — or, stated more appropriately, the blending of functions — that are given for and associated with sites (or regions) in a D-dimensional domain. The figure (above) illustrates two locally defined functions $\varrho_1(X)$ and $\varrho_2(X)$ with associated sites X_1 and X_2 that must be blended. The figure also includes weight / blending functions $w_1(X)$ and $w_2(X)$ that must be constructed and used for blending such that $\varrho_1(X)$ and $\varrho_2(X)$ are interpolated / reproduced exactly at the sites X_1 and X_2, respectively. Shepard's method can serve as an example to demonstrate the principle: Given N sites $\mathbb{X}_i$ with associated function values f_i, employ an "inverse-squared-distance" blending for interpolation, i.e., define

$$f(\mathbb{X}) = \begin{cases} \sum_i {}^1/_{d_i^2} \cdot f_i \;/\; \sum_i {}^1/_{d_i^2} & , \mathbb{X} \neq \mathbb{X}_i \\ f_i & , \mathbb{X} = \mathbb{X}_i \end{cases},$$

where $d_i^2 = d_i^2(\mathbb{X}) = \|\mathbb{X} - \mathbb{X}_i\|^2 = (x_1 - x_i^1)^2 + \ldots + (x_D - x_i^D)^2$.

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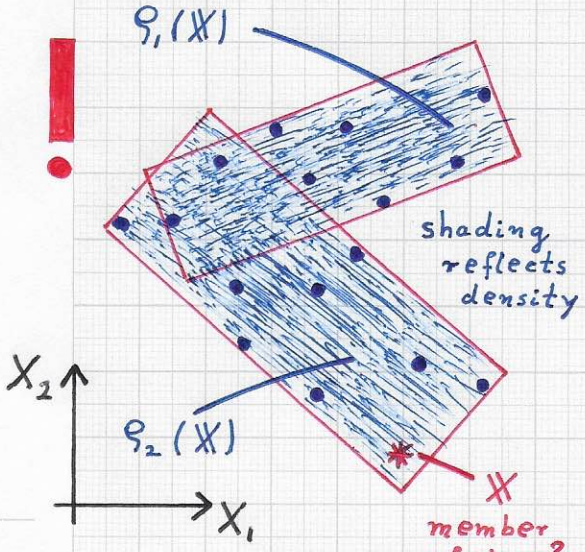
OBJECT AND MATERIAL EIGENFUNCTIONS - Cont'd.

• Laplacian eigenfunctions and neural networks: ...



The left figure illustrates the scattered data – where data must be understood as functions – interpolation problem that we must devise a solution for. The figure shows a scenario where the distribution of a finite point set $\{\bullet\}$ representing a specific class cannot be bounded "tightly" by just one minimal bounding box. Thus, splitting is applied to the point set (and resulting subsets) until a set of minimal (local) bounding boxes is obtained, where all boxes satisfy some specified criteria concerning box geometry and point distribution in a box. Further, we have described an approach that maps a box to the square region $[-1,1]^2$, in the sketched 2-dimensional scenario, and vice versa. Based on the discrete point distributions in all boxes, approximation functions – "probability density functions" – can be constructed for all point subsets associated with certain boxes. In the figure, these functions are called $\varphi_1(\mathbb{X})$ and $\varphi_2(\mathbb{X})$. For example, they could be constructed via least-squares methods applied to a (low-degree) polynomial or some "standard" point distribution model.

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■ OBJECT AND MATERIAL EIGENFUNCTIONS - Cont'd.

• Laplacian eigenfunctions and neural networks:...

• Note. Number of points, number of dimensions. The number of points representing a specific (material) should be understood as a relatively "small number," reaching 100000 only seldom — since this number is the number of real, physical objects serving as sample objects. The number of dimensions (D) used to perform the described statistical data representation and classification computations and comparisons should also be viewed (or "defined") as a relatively "small number"; the formulae provided on p. 9 (4/25/2023) show that the volume of a D-ball with radius R is proportional to R^D, and, therefore, performing classification in a rather high-dimensional space ($D > 10$, for example) would be difficult, or even unreliable, considering the fact that an already small number of class sample points implies low point densities in D-space. (Reference: "Smooth interpolation of scattered data by local thin plate splines," Comp. & Maths. with Appls. 8(4), pp. 273-281, by Richard Franke.) This paper describes approaches that can be adapted to be applicable to our goal, i.e., approximating a multivariate probability density function $\rho(\mathbf{X})$ by blending "simple" local density function approximations $\rho_i(\mathbf{X})$ via blending/weight functions $w_i(\mathbf{X})$:

$$\rho(\mathbf{X}) = \sum_i w_i(\mathbf{X})\, \rho_i(\mathbf{X}).$$

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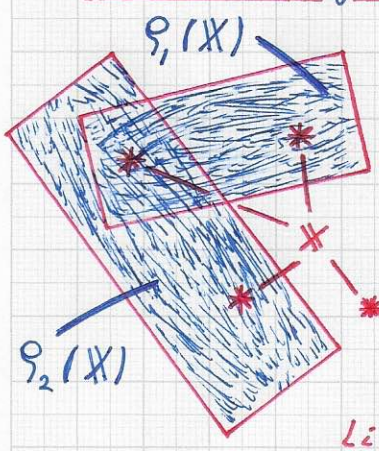
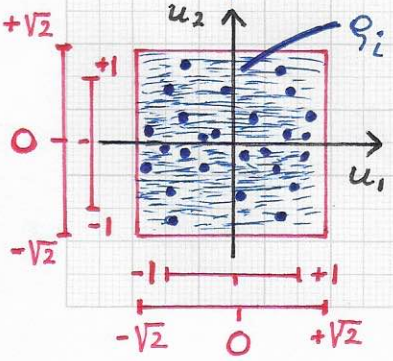
OBJECT AND MATERIAL EIGENFUNCTIONS - Cont'd.

- Laplacian eigenfunctions and neural networks: ... (Reference: "A generalization of algebraic surface drawing," ACM Transactions on Computer Graphics 1(3), pp. 235-256, by James F. Blinn.) In the referenced paper by Blinn, the concept of "metaballs" was introduced, where a set of trivariate density functions is used to model sphere-like, implicitly defined surfaces that are blended together (pp. 237-239 of referenced paper).



The left figure shows a simple scenario, driven by our classification problem: The point * to be classified lies outside the two "domain boxes" of density functions ϱ_1 and ϱ_2; lies inside only one or inside both "domain boxes." When * lies inside the overlap region of the two boxes, one will have to blend the probability values $\varrho_1(X)$ and $\varrho_2(X)$, e.g., $\varrho(X) = \left(\sum_{i=1}^{2} \frac{1}{(d_i)^2}\, \varrho_i(X)\right) \Big/ \left(\sum_{i=1}^{2} \frac{1}{(d_i)^2}\right)$, where $(d_i)^2 = \|X - \bar{\Phi}_i\|^2$ and $\bar{\Phi}_i$ is the center of box i. (See p. 22, 5/16/2023.) Once class points (•) have been mapped to a corresponding normalized and bounded domain (left figure), one must determine, via a least-squares method, the parameter values of a "proper model," e.g., $\varrho_i(u_1, u_2) = A_i \cdot \exp\left(-(B_i u_1^2 + C_i u_1 u_2 + D_i u_2^2)\right)$.



These concepts must be adapted to the D-dimensional case. ...