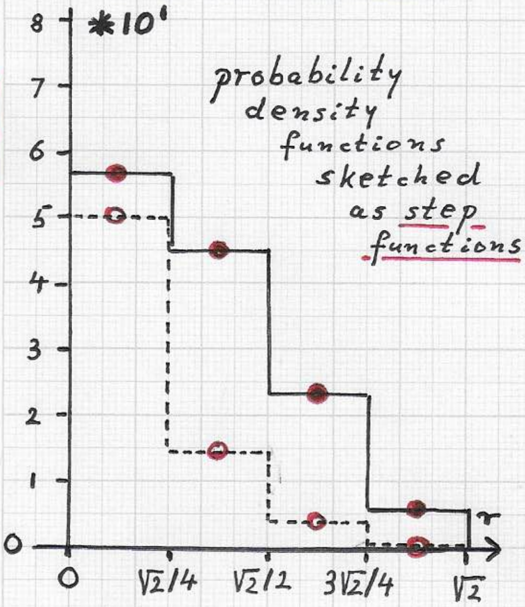
OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

- Laplacian eigenfunctions and neural networks:…



Thus, one obtains the value of ${}^2\varrho_i$ in terms of known/calculated ϱ_i-values as

$${}^2\varrho_i = \varrho_i \frac{2}{(2i+1)\pi} \cdot \sqrt{2} .$$

We specify the observed values of n_i as $n_0 = 20$, $n_1 = 16$, $n_2 = 8$ and $n_3 = 2$. Since the spacing on the r-axis is $\sqrt{2}/4$, the ϱ_i-values are $n_i/(\sqrt{2}/4)$. The values of ${}^2\varrho_i$ are defined as ${}^2\varrho_0 = \varrho_0 \cdot 2/\pi \sqrt{2}$, ${}^2\varrho_1 = \varrho_1 \cdot 2/(3\pi) \cdot \sqrt{2}$, ${}^2\varrho_2 = \varrho_2 \cdot 2/(5\pi) \cdot \sqrt{2}$ and ${}^2\varrho_3 = \varrho_3 \cdot 2/(7\pi) \cdot \sqrt{2}$. The left table lists the values involved. The top-left figure provides a sketch, where ϱ_i-values are shown as '●' and ${}^2\varrho_i$-values as '○'.

i	n_i	ϱ_i	${}^2\varrho_i$
0	20	56.56	50.93
1	16	45.25	13.58
2	8	22.63	4.07
3	2	5.66	0.78

- **Note.** Given a w-tuple in normalized w-space, we must determine a probability value for this point defining the likelihood that the w-tuple represents the (material) class associated with the $\sum n_i$ tuples that satisfy $0 \le \|w\|^2 \le D$. Using the n_i-values defined above and ignoring D, a simple probability is defined as $p_i = n_i / \sum n_i = n_i/146$, i.e., $p_0 = 20/146$, $p_1 = 16/146$, $p_2 = 8/146$ and $p_3 = 2/146$.

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Bernd Hamann
Stratovan

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

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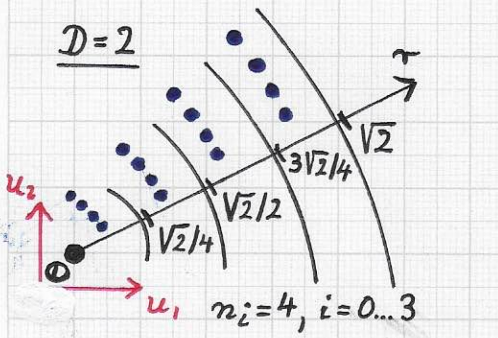
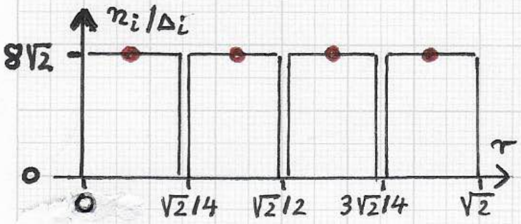


i	R_i	Δ_i	n_i	n_i/Δ_i
0	0	$\sqrt{2}/4$	4	$8\sqrt{2}$
1	$\sqrt{2}/4$	$\sqrt{2}/4$	4	$8\sqrt{2}$
2	$\sqrt{2}/2$	$\sqrt{2}/4$	4	$8\sqrt{2}$
3	$3\sqrt{2}/4$	$\sqrt{2}/4$	4	$8\sqrt{2}$
4	$\sqrt{2}$	/	/	/



$$\|n_i/\Delta_i\| = \left(\int_0^{\sqrt{2}} (8\sqrt{2})^2 dr\right)^{1/2}$$
$$= (128\sqrt{2})^{1/2}$$
$$= \ldots = 8\cdot 2^{3/4}$$

⇒ normalization: divide n_i/Δ_i-values by $(8\cdot 2^{3/4})$:

$$(8\sqrt{2}):(8\cdot 2^{3/4}) = \ldots = 2^{-1/4}$$

$$\Rightarrow \left(\int_0^{\sqrt{2}} (2^{-1/4})^2 dr\right)^{1/2} = 1,$$

where $2^{-1/4} \approx 0.84$.

We consider the issue of data **NORMALIZATION** in more detail via a simple example for $D=2$, see left figure. This issue is crucially important for the proper probabilistic classification of a u-tuple. For simplicity, we use four layers having the same "thickness" $(\Delta = \Delta_i = R_{i+1} - R_i = \sqrt{2}/4)$, see left table. Further, we assume that there are four points '•' in each layer $(n = n_i = 4)$. Thus, an "initial density function" can be defined for $[0, \sqrt{2}]$ with constant values n_i/Δ_i per interval $[R_i, R_{i+1})$; here, $n_i/\Delta_i = 8\sqrt{2}$. The resulting step function is shown in the left figure. Next, we perform normalization via the L_2-norm. The first normalization steps generates the constant value $2^{-1/4} \approx 0.84$ for the entire domain on the r-line, i.e., $[0, \sqrt{2}]$, see the calculation (left). •••

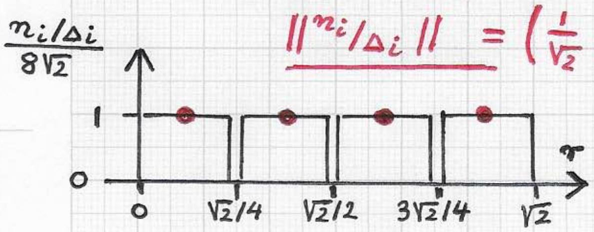
■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions and neural networks ... For this example, normalization should actually produce the constant value 1 for the r-line domain $[0, \sqrt{2}]$. We achieve this normalization goal by using

$$\|n_i/\Delta_i\| = \left(\frac{1}{\sqrt{2}} \int_0^{\sqrt{2}} (8\sqrt{2})^2 \, dr\right)^{1/2}$$

as norm, where the integral value is divided by the length of the domain interval, i.e., $\sqrt{2}$. The value of this norm for this example is

$$\|n_i/\Delta_i\| = \left(\frac{1}{\sqrt{2}} \int_0^{\sqrt{2}} 128 \, dr\right)^{1/2} = \ldots = 8\sqrt{2}.$$



Dividing the n_i/Δ_i-values $(8\sqrt{2})$ by the value of the norm $(8\sqrt{2})$ produces the normalized function 1 for the interval $[0, \sqrt{2}]$, see figure. (The norm-value of this function is, as desired by design, $\left(1/\sqrt{2} \int_0^{\sqrt{2}} 1 \, dr\right)^{1/2} = 1$.)

i	R_i	Δ_i	n_i	n_i/Δ_i
0	0	$\sqrt{2}/4$	4	$8\sqrt{2}$
1	$\sqrt{2}/4$	$\sqrt{2}/4$	3	$6\sqrt{2}$
2	$\sqrt{2}/2$	$\sqrt{2}/4$	2	$4\sqrt{2}$
3	$3\sqrt{2}/4$	$\sqrt{2}/4$	1	$2\sqrt{2}$
4	$\sqrt{2}$	/	/	/

In our next example, we use the data listed in the left table. Here, the n_i-values decrease linearly from 4 to 1 with increasing layer index i. The value of n_i/Δ_i is given by $n_i/\Delta_i = 2 n_i \sqrt{2}$. Thus, we obtain

$$\|n_i/\Delta_i\| = \left(\frac{1}{\sqrt{2}}\left[\int_0^{\sqrt{2}/4} (8\sqrt{2})^2 dr + \int_{\sqrt{2}/4}^{\sqrt{2}/2} (6\sqrt{2})^2 dr + \int_{\sqrt{2}/2}^{3\sqrt{2}/4} (4\sqrt{2})^2 dr + \int_{3\sqrt{2}/4}^{\sqrt{2}} (2\sqrt{2})^2 dr\right]\right)^{1/2} = \ldots = \left(\frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{4} \cdot [128 + 72 + 32 + 8]\right)^{1/2}$$

$$= \left(\frac{1}{4} \cdot 240\right)^{1/2} = \ldots = 2\sqrt{15}.$$

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■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

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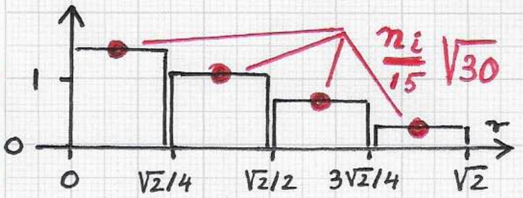
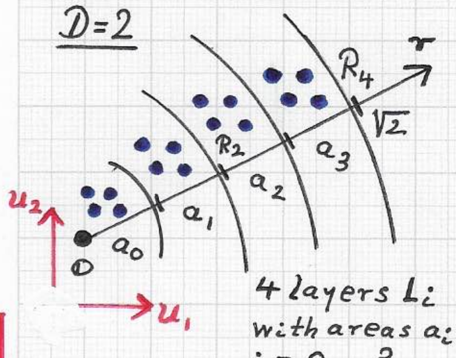


We divide the n_i/Δ_i-values ($8\sqrt{2}$, $6\sqrt{2}$, $4\sqrt{2}$ and $2\sqrt{2}$) by $2\sqrt{15}$ to produce the desired normalized function for the interval $[0, \sqrt{2}]$, see left figure.

The resulting values are $(2 n_i \sqrt{2}) : (2\sqrt{15}) = \frac{n_i}{15}\sqrt{30}$, $i = 0 \ldots 3$. These linearly decreasing values are indicated in the top-left figure as '•'; the approximate values are 1.46, 1.10, 0.73 and 0.37. (The norm-value of this function is, as desired,

$$\left(\frac{1}{\sqrt{2}}\left[\int_0^{\sqrt{2}/4} (4/_{15}\sqrt{30})^2\,dr + \int_{\sqrt{2}/4}^{\sqrt{2}/2} (3/_{15}\sqrt{30})^2\,dr + \int_{\sqrt{2}/2}^{3\sqrt{2}/4} (2/_{15}\sqrt{30})^2\,dr + \int_{3\sqrt{2}/4}^{\sqrt{2}} (1/_{15}\sqrt{30})^2\,dr\right]\right)^{1/2} = \ldots = 1.)$$



We now perform normalization relative to actual layers and layer areas ($D=2$), i.e., we consider the density of points '•' in the D-manifold layers they lie in. The left figure and table illustrate this case for $D=2$ and provide a specific numerical example. Again, we use the equidistant spacing $R_i = i\sqrt{2}/4$ and constant point numbers $n_i = 4$, $i = 0 \ldots 3(4)$. The

i	n_i	a_i	n_i/a_i
0	4	$\frac{1}{8}\pi$	$32/\pi \approx 10.19$
1	4	$\frac{3}{8}\pi$	$32/(3\pi) \approx 3.40$
2	4	$\frac{5}{8}\pi$	$32/(5\pi) \approx 2.04$
3	4	$\frac{7}{8}\pi$	$32/(7\pi) \approx 1.46$

layer hyper-volumes (areas) are $a_i = \pi(R_{i+1}^2 - R_i^2) = \ldots = \ldots = \pi \cdot (2i+1)/8$, $i = 0 \ldots 3$, see table. • • •

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

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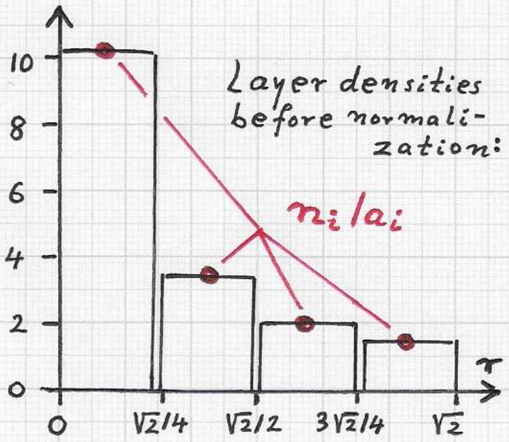

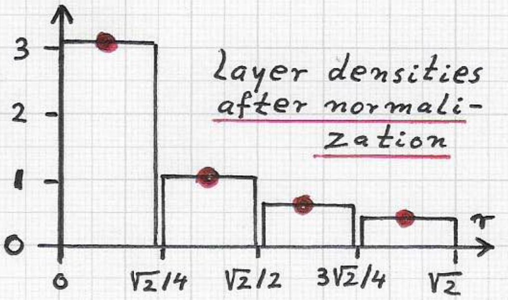

i	n_i/a_i	$\sqrt{105/11}/(2i+1)$
0	$32/\pi$	$\sqrt{105/11} \approx 3.09$
1	$32/(3\pi)$	$\frac{1}{3}\sqrt{105/11} \approx 1.03$
2	$32/(5\pi)$	$\frac{1}{5}\sqrt{105/11} \approx 0.62$
3	$32/(7\pi)$	$\frac{1}{7}\sqrt{105/11} \approx 0.44$

The table (previous page) and the figure (left) show the initial values of n_i/a_i before normalization. They are defined as $n_i/a_i = 4 : (\pi \cdot (2i+1)/8) = 32/((2i+1)\pi)$. The left figure plots these density values over the one-dimensional r-line – but the density values are relative to **area!** To obtain the wanted probability density values normalized relative to the two-dimensional domain, we must perform integrations for areas. The total domain area is $\pi(\sqrt{2})^2 = 2\pi$; thus, we must divide the integral value of the squared layer densities by 2π. The left table shows the final values of layer probability densities after normalization (last, third column).

We must calculate the "root-mean-squared metric" $\|\cdot\|$:
$\|\cdot\| = \left(\frac{1}{2\pi}\int_\Omega (\text{layer\_densities})^2\, da\right)^{1/2}$, where Ω is the total area for integration, i.e., the **2**-ball with area 2π. ...